Answers to EAS 327 Trial Problems: Last Updated 1 April, 2004

1. Since the resistors are identical, the output voltage (proviso: neglecting the source resistance, which there is no way to know from the given information) is half the supply voltage, ie. $V_0 = 3$ volts. The current is $i = 6/(2\ K\Omega) = 3$ milliamps.

2. Again, $V_0 = 3$ volts, but the current is far smaller at $i = 6/(2 \text{ x } 10K\Omega) =$ 0.3 milliamps

3. Since $2 = 6\ R_2/(R_1 + R_2)$ and $R_2 = 10\ K\Omega$, by rearrangement $R_1 =$ $20\ K\Omega$

4. Let's take the case that $T = 10^o$ C. Then the element R_1 has resistance $R_1 = 10K\ \ (1 + 0.00392 * (10 - 20)) = 9.608K$ (I shall often drop the "Ω" and a "K" means a kilohm). So the output voltage, when the temperature is 10^o C, is (given to three significant figures)

$$V_0 = 12\ \frac{10}{10 + 9.608} = 6.12 \text{ volts}$$

5. The RC time-constant $\tau = RC = 5 \text{ x } 10^3 \text{ x } 1 \text{ x } 10^-6 = 5$ millisec. The half-power frequency is thus $f_0 = 1000/10\pi = 31.8$ Hz.

   However in fact one does not need to know the half-power frequency to answer the question... recall that

$$G(f) = \frac{1}{1 + \left(\frac{f}{f_0}\right)^2}$$

and so if $f = 0$, $G = 1$; if $f = f_0$, $G = \frac{1}{2}$; and if $f = f_0/2$, $G = \frac{1}{1+0.25} = 0.8$

6. At $T = 25^o$ C the sensing element has resistance $R_1 = 20K(1+0.04(25-20)) = 24K$. Then at this temperature $V_0 = 5\ 20/(20+24) = 2.27$ volts

7. At $T = 20^o$ C the sensing element has resistance $R_1 = 20K(1+0.04(20-20)) = 20K$ (obviously!). So the output voltage is $V_0 = 10\ 5/(5+20) = 2.0$ volts.

   At $T = 20.05^o$ C the sensing element has resistance $R_1 = 20K(1 + 0.04(20.05 - 20)) = 20.04K$. Then $V_0 = 10\ 5/(5 + 20.04) = 1.99681$ volts. This signal is only 3.19 millivolts different from the signal at 20 C, and lies outside the input span (ie. off scale) for a receiver with full scale range (FSR) ± 0.005 volts.

   If the datalogger has FSR ± 10 volts and 12 bits resolution, the minimum detectable change (ie. voltage resolution) of the logger is

$$\delta_V = \frac{20}{2^{12} - 1} = 4.884 \text{ x } 10^{-3} \text{volts}$$

   ie. the mimimum detectable change is 4.88 millivolts. Thus although the FSR of ± 10 volts will permit to measure the voltage out of our half-bridge temperature sensor, the 12 bit resolution will not permit to detect a temperature shift as small as 0.05 C.

   So we subtract off 2.0 volts, by introducing the second arm of the bridge (ie. work with a full bridge circuit). Now, at 20.05 C, we have a bridge error voltage of -3.19 millivolts, small enough that we can work with a receiver whose FSR is ± 5 millivolts.

The voltage resolution with this setup is

$$\delta_V = \frac{10}{2^{14} - 1} = 0.610 \text{ x } 10^{-3} \text{ millivolts}$$

or 0.61 microvolts ($\delta_V = 0.61\ \mu V$). What is the corresponding temperature resolution? Well we have computed that a shift in temperature of 0.05 C causes a shift in the output voltage of 3190 μV, so our temperature resolution is

$$\delta_T = \delta_V \, \frac{0.05}{3190} = 0.61 \, \frac{0.05}{3190} = 9.56 \text{ x } 10^{-6} \text{ K}$$

8. Only the ratio N_L/N_W is "dimensionless"

9. $N'_L = 2N_L$ and , $N'_W = 2N_W$, but N'_L/N'_W is identical to N_L/N_W

10. Conductivity of copper is $k = 385$ W m^{-1} K^{-1} so the magnitude of the heat flux density is

$$|Q_H| = k \, \frac{20}{1} = \ 7.7 \text{ x } 10^3 \text{ W m}^{-2}$$

11. Because the temperature T of the sphere is not changing, we have

$$C \, \frac{dT}{dt} = 0 = A \, (Q^* + Q_H) + P$$

where I have assumed the sphere is dry. The left hand side is the term that involves the heat capacity of the sphere, and it is zero. Thus the power P provided to the hot sphere is lost to the environment as sensible heat and in the form of radiation.

Assume only longwave radiation needs to be considered, then to a first approximation

$$Q^* = \sigma\left(T^4 - T_a^4\right) = 5.67 \text{ x } 10^-8\ \left((273+850)^4 - (273+25)^4\right) = 8.97 \text{ x } 10^4 \text{ W m}^{-2}$$

which is a very large energy flux density, but of course the surface area of the sphere is small, $A = \pi(0.0015)^2 = 2.43 \text{ x } 10^{-5} \text{ m}^2$

What about the sensible heat flux density? The temperature difference is large ($\Delta T = 825^o$ C). We are told the air is still, so we need the Nusselt number for a sphere in still air. This is given in the course tables as

$$Nu = 2 + 0.589 R_a^{1/4}\left[1 + \left(\frac{0.469}{P_r}\right)^{9/16}\right]^{-4/9} \tag{9}$$

where the Rayleigh number $R_a = G_r P_r$ and the Prandtl number $P_r = \nu/\kappa$ is the ratio of the fluid's kinematic viscosity to its thermal diffusivity. For air at 25^o C, $P_r = 1.55/2.22 = 0.698$ (see tables).

Putting in the numbers, in our problem the Grashof number

$$G_r = \frac{g\ d^3\ \Delta T}{273\ \nu^2} = \frac{9.81\ 0.0015^3\ 825}{273\ \left(1.55\text{x}10^{-5}\right)^2} = 416$$

Thus $R_a = 0.698 * 416 = 291$ and we can compute the Nusselt number from equation (9), $N_u = 3.87$.

Now we deduce the heat transfer resistance

$$r_H = \frac{d}{\kappa\ N_u} = \frac{0.0015}{2.22\text{x}10^{-5}\quad 3.87} = 17.4 \text{ s m}^{-1}$$

and finally our sensible heat flux density is

$$Q_H = \rho\, c_p\, \frac{\Delta T}{r_H} \approx 10^3\, \frac{825}{17.4} = 4.74 \text{ x } 10^4 \text{ W m}^{-2}$$

(I assumed $\rho \approx 1$ kg m^{-3}) which is again, a very large heat flux density.

So the power required to heat this sphere is

$$P = A\,(Q^* + Q_H) = 2.43\text{x}10^{-5}\text{x}(8.97 + 4.74)\text{x}10^4 = 3.3 \text{ W}$$

You deserve a coffee after that one!

12. This follows from the example done in class

13. Suppose it happens that air temperature is $T = 25$ C, and wet-bulb temperature is $T_w = 20$ C. Suppose a wet-bulb thermometer, radiation-shielded, and exposed to a ventilating draft of speed $u = 4$ m s^{-1}, is in equilibrium at the wet-bulb temperature (ie. it is performing ideally). Its wick is cylindrical, with diameter $d = 4$ mm, and with length $\ell = 2$ cm. The water reservoir contains 5 cm^3. Calculate the interval of time required to empty the reservoir.

    There are many ways to approach this problem. The simplest is as follows. We need the evaporation rate $E = Q_E/L_v$ and we know that

$$Q_E = \frac{\rho\, c_p}{\gamma}\, \frac{e_*(T_w) - e}{r_v} \tag{14}$$

    but for the ideal wet bulb thermometer $|Q_E| = |Q_H|$ so it will be simpler to compute Q_H. This is easy.

$$|Q_E| = |Q_H| = \rho c_p\, \frac{T - T_w}{r_H} \tag{15}$$

We need the density: I didn't give you the pressure (which, by the way, you would have needed in order to evaluate the psychrometric constant $\gamma = pc_p/(0.622L_v)$), so lets put $\rho c_p = 10^3$ (MKS units). All we need now is r_H. This we can get from the Nusselt number for a cylinder in forced convection...

The Reynolds number $R_e = ud/\nu = 1067$, so $N_u = 0.24\ R_e^{0.6} = 16$ (two significant figures is sufficient). Then $r_H = d/(N_u\ D_H) = 12\ s\ m^{-1}$ (to get these results I assumed $\nu = 1.5$ x 10^{-5} m^2 s^{-1}, $D_H = 2.1$ x 10^{-5} m^2 s^{-1}).

So now we can compute that $|Q_E| = 10^3 * 5/12 = 420$ W m^{-2} and since the latent heat of vaporization $L_v = 2.5$ x 10^6 J kg^{-1} the evaporation rate is 1.7 x 10$^-$4 kg m^{-2} s^{-1}. Multiply this by the surface area ($\pi d\ell = 2.5$ x 10^{-4} m^2) to deduce that the rate of loss of mass from the reservoir is 4.3 x 10^{-8} kg s^{-1}.

Initially the mass of water is the reservoir is $M_0 = 1000$ kg m^{-3} x 5 x $(10^{-2}\ \mathrm{m})^3 = 5$ x 10^{-3} kg. So it will take 5 x 10^{-3}/ (4.3 x 10^{-8}) = 1.2 x 10^5 seconds ($\boxed{\text{1.4 days}}$) to drain the reservoir.

14. Suppose a pitot tube is aligned with a unidirectional airflow of velocity U and density $\rho = 1$ kg m^{-3}. The pressure difference Δp between the stagnation and static ports is relayed to a manometer, which is filled with water. Calculate the "signal" h (height of the displaced column of water) if $U = 0.1$ m s^{-1} (etc).

The pressure difference $\Delta p = \frac{1}{2}\rho U^2 = 5 \text{ x } 10^{-3}$ Pa (assuming $\rho = 1$ kg m^{-3}) and this will cause a signal $\rho_w g h = 10^4 h = 5 \text{ x } 10^{-3}$ or $h = 5 \text{ x } 10^{-7}$ m, ie. $\boxed{h = 0.5\ \mu m}$.

15. Now suppose the same pitot tube/manometer reports $h = 15$ mm, when aligned to a unidirectional flow of water. What is the water velocity U?

    We have $\rho_w g \text{ x } 15 \text{ x } 10^{-3} = \frac{1}{2}\rho_w U^2$. The water density cancels and $U^2 = 2g \text{ x } 15 \text{ x } 10^{-3}$, so $\boxed{U = 0.55 \text{ m s}^{-1}}$.

16. Suppose a cup anemometer, sitting in an airstream with speed $s = 5$ m s^{-1} and density $\rho = 1$ kg m^{-3}, is restrained from rotating. Calculate the drag torque Γ on the anemometer if: frontal area of the cups $A = 25$ cm^2; radius to centre of cups $r = 10$ cm; and drag coefficients $c_{df} = 1,\ c_{db} = 0.5$.

    $\Gamma = r \text{ x } \rho\, s^2\, A\, (c_{df} - c_{db}) = 3.1 \text{ x } 10^{-3}$ Nm.

17. Suppose x is uniformly distributed on the range $-1 \leq x \leq 1$. Then the p.d.f. of x is:

$$\begin{aligned} f(x) &= 0.5, \quad |x| \leq 1 \\ &= 0, \quad |x| > 1 \end{aligned} \tag{16}$$

    Calculate $E[x]$, $E[x^2]$, and the standard deviation σ_x.

    Since the pdf is symmetric about $x = 0$ the mean $E[x]$ must be zero.

This is easy to prove:

$$E[x] = \int_{-\infty}^{\infty} x\ f(x)\ dx = \int_{-1}^{1} x\ dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0 \tag{17}$$

Since the mean is zero, the deviation $x' = x - E[x] \equiv x$ and so the variance

$$E[x'^2] = \int_{-\infty}^{\infty} x^2\ f(x)\ dx = \int_{-1}^{1} x^2\ dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3} \tag{18}$$

and thus $\sigma_x = \sqrt{2/3}$

18. Suppose x can take on values $0 \le x \le \infty$, and that x has p.d.f.

$$f(x) = \alpha\ \exp\left(-\ \alpha x\right) \tag{19}$$

- The units of α must be the reciprocal of the units of x
- Is this pdf normalized? Check:

$$\int_{0}^{\infty} f(x)\ dx = \alpha\ \left[\frac{e^{-\alpha x}}{-\alpha}\right]_{0}^{\infty} = \alpha\ \left[0 - \frac{1}{-\alpha}\right] = 1 \tag{20}$$

  Yes, it is normalised.

- Calculate $E[x]$, $E[x^2]$, and the standard deviation σ_x

To obtain the mean, one needs to do an integration by parts:

$$\begin{aligned} E[x] &= \int_0^\infty x\, f(x)\, dx \\ &= \alpha \left(\left[x \, \frac{e^{-\alpha x}}{-\alpha} \right]_0^\infty - \int_0^\infty \frac{e^{-\alpha x}}{-\alpha}\, dx \right) \\ &= \int_0^\infty e^{-\alpha x}\, dx \\ &= \left[\frac{e^{-\alpha x}}{-\alpha} \right]_0^\infty \\ &= \frac{1}{\alpha} \end{aligned} \tag{21}$$

Thus the parameter α is the reciprocal of the mean. To get the mean square or the variance, one will need to integrate by parts twice (a good exercise if you want to practise your Calculus, but otherwise don't worry!). The standard deviation turns out to be equal to the mean.

- If the parameter $\alpha = 1$, find the probability P that a random realisation (choice) of x lies in the range $3 \leq x \leq 4$.

  Here we want

$$P = \int_3^4 e^{-x}\, dx = \left[\frac{e^{-x}}{-1} \right]_3^4 = \left[e^{-3} - e^{-4} \right] = 0.0315 \tag{22}$$

19. Prove by substitution that the function $\beta_v = B\,[1 - \cos(t/T)]$ satisfies the differential equation

$$\frac{d^2\beta_v}{dt^2} = \frac{\beta - \beta_v}{T^2} \tag{23}$$

where β is a constant.

First, note by differentiation that

$$
\begin{aligned}
\frac{d\beta_v}{dt} &= B(-1)\frac{d\cos(t/T)}{dt} = B(-1)\frac{-1}{T}\sin(t/T) = \frac{B}{T}\sin(t/T) \\
\frac{d^2\beta_v}{dt^2} &= \frac{d}{dt}\left(\frac{d\beta_v}{dt}\right) = \frac{d}{dt}\left(\frac{B}{T}\sin(t/T)\right) \\
&= \frac{B}{T}\,\frac{1}{T}\,\cos(t/T) = \frac{B}{T^2}\cos(t/T)
\end{aligned}
\tag{24}
$$

Thus

$$
\begin{aligned}
\text{LHS} &= \frac{d^2\beta_v}{dt^2} = \frac{B}{T^2}\cos(t/T) \\
\text{RHS} &= \frac{\beta - B\,[1 - \cos(t/T)]}{T^2} \\
&= \frac{\beta - B}{T^2} + \frac{B\cos(t/T)}{T^2} \\
&= \frac{\beta - B}{T^2} + \text{LHS}
\end{aligned}
\tag{25}
$$

so LHS $\equiv$ RHS provided $B = \beta$.