Numerical Forecasting with the Barotropic Model[1]

By BERT BOLIN, University of Stockholm

(Manuscript received January 20, 1955)

Abstract

The experiments with numerical forecasting using the barotropic model have been continued and a number of 24, 48 and 72 hour forecasts are presented. The initial data for these forecasts covered an area of about 9,000 by 12,000 km. The results indicate that the boundary influences have been reduced to be unimportant in the centre of the area in the 24-hour forecasts but they may still cause errors in the 72-hour forecasts. In view of the approximations made very good 72-hour forecasts have been obtained in some cases. The most successful one gave a correlation of 0.87 and a relative error of the forecast height changes of 0.58. Most of these extended forecasts should be of definite value in forecasting the weather. Large errors are, however, still obtained in some cases. The neglection of baroclinic process is one of the reasons for these errors but it is quite obvious that the largest errors are of a different nature. Experiments are at present conducted to find the sources of these errors. One line of this research is a general re-analysis of the assumptions made in deriving the barotropic model which is presented in the latter part of this paper. A number of improvements are suggested for further tests.

1. Introduction

In a recent report (STAFF MEMBERS, INSTITUTE OF METEOROLOGY, UNIVERSITY OF STOCKHOLM, 1954, hereafter denoted by II) a number of 24-hour forecasts with the aid of the barotropic model were presented. In spite of the comparatively successful results it was concluded that several improvements in the computational procedure were desirable. In particular the assumptions on the boundaries of the forecast area affected the forecasts seriously in some cases. It was not possible to use a larger area than about 5,700 × 5,700 km with a gridsize of 300 km, because of the limited capacity of the Swedish Computer BESK at that time. Nor was it possible to extend the forecasts beyond 24 hours for the same reason. Since then a magnetic drum capable of holding 4,096 40 binary digit words has been added to the machine, which has made it possible to extend the forecasts to cover an area more than three times as large as the one previously used and also accordingly increase the forecast period. A number of such forecasts have now been made and the results of those are presented here.

The first series of such forecasts over two and three days immediately made it clear that it was highly desirable to make such tests over a longer period both from the point of view of using them in daily weather forecasting routine as well as for the analysis of different kinds of errors. Such an extensive testing program therefore was started by the Weather Service of the Swedish Air Force. Since December 1, 1954 more or less regular 24, 48 and 72-hour forecasts are made. As a com-

[1] The research reported in this document has been sponsored in part by the Geophysics Research Directorate of the Air Force Combridge Research Center, *Air Research and Development Command, United States Air Force*, under contract No. AF 61 (514)-648-C, through the European Office, ARDC.

parison the Swedish Weather Bureau prepares forecasts with conventional extrapolation methods. Some of the preliminary results of these forecasts are considered in the discussion here.

By and large the results are very encouraging and even the forecasts over 72 hours ought to be useful in actual weather forecasting in most cases. In the centre of the forecast area the effects of the assumptions on the boundaries seem to be small. However, many systematic errors still appear as can be seen from the presentation below. In a few cases these errors are obviously due to the fact that we disregard baroclinic effects completely, but in most cases there seem to be other reasons for the disagreement between the computed and the real development. Among other things it therefore seems very desirable to reconsider the approximations within the barotropic model and analyze the effects of them. Such an analysis is attempted in the latter part of this paper.

FJØRTOFT (1952) has developed a graphical method for solving the barotropic vorticity equation. Since this method is very simple and can be used in the daily forecasting routine in any weather service without having an electronic computer at disposal it is of very definite interest to compare the results obtained in such a way with the results of computations on a machine. Such a comparative study has been made by O. Haug and is also presented here.

The initiation of this work has been possible by grants from the Knut and Alice Wallenberg Foundation and the Swedish National Science Research Council and by the support from the Office of Naval Research, USA. The continuation has been supported by the Weather Service of the Swedish Air Force and the Air Research and Development Command of the United States Air Force. The practical work of carrying through the forecasting program discussed here was done by a team consisting of H. Bedient, USA; P. Bergthorsson, Iceland; G. Dahlquist, Sweden; B. Döös, Sweden; O. Haug, Norway, A. Jonasson, Sweden, and the author. A number of discussions with Drs. P. Welander and G. Dahlquist have contributed to the general discussion in the latter part of the paper. All these contributions are gratefully acknowledged. Finally I am greatly indebted to Professor C.-G. Rossby for his neverfailing interest in this work and for many stimulating discussions.

2. Methods of computations and truncation errors

In the attempts to use the barotropic model over a larger area and over periods of two or three days about the same procedure at first has been used as given by CHARNEY and PHILLIPS (1953) and which also was used in II. A few minor changes have been made:

The criterion for computational stability puts an upper limit on the length of the time-steps, which is about one hour if the gridsize is 300 km. The reason for this is that only the neighbouring gridpoints are used when computing the change in time at a point with the aid of finite differences in space. It is quite obvious that the time derivative thus obtained only can be representative for a time period which is equal to or less than the time it takes for influences to travel the distance between these neighbouring grid points and the point itself. To be able to take longer time steps it is necessary to use a method by which influences from points at larger distances automatically are included if necessary. The following method has been tried: In order to compute the change of vorticity at a certain point, (x_0, y_0) from the time t to $t+\Delta t$, we try to find the location (x', y') of a particle at time t, which at time $t+\Delta t$ will be located at (x_0, y_0). The difference between the vorticities in the two points (x', y') and (x_0, y_0) is then equal to the vorticity change in (x_0, y_0) over the time interval Δt. The size of Δt is here limited by the accuracy with which the coordinates (x', y') can be determined. By using second differences in the interpolation schemes it has been possible to use time steps of 3 hours in a field where the maximum velocity of the flow was 60 m/sec. It was believed that this increase of the time step would shorten the total time for a forecast but the computational method itself is considerably more complicated than a Jacobian computation. Furthermore the points close to the boundary require a special handling which still more increases the length of the computations. No gain in time was thus obtained with this method. Instead the following scheme was used in the final computations (devised by G. Dahlquist).

The basis of the method for the integration of the equations

$$\frac{\partial \xi}{\partial t} = \frac{\partial \phi}{\partial y} \cdot \frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial x} \cdot \frac{\partial \eta}{\partial y} = J(\eta, \phi) \qquad (2.1)$$

$$\eta = \frac{m^2}{f} \xi + f \qquad (2.2)$$

$$\xi = \nabla^2 \phi \qquad (2.3)$$

is the trapezoidal rule

$$\xi^\tau - \xi^{\tau-1} = \Delta t \cdot J\left(\frac{\eta^\tau + \eta^{\tau-1}}{2}, \ \phi^{\tau - 1/2}\right) \qquad (2.4)$$

and the usual finite difference approximation

$$\xi^\tau = (\bar{\phi}^\tau - \phi^\tau) \cdot \frac{4}{(\Delta s)^2} \qquad (2.5)$$

Here ξ^τ etc. denotes the value of ξ at the time $\tau \cdot \Delta t$, the bar denotes the average of ϕ^τ in the four neighbouring lattice-points and Δs is the mesh-width (which is equal to 300 km at 50° latitude). Other notations as in (II). Since ξ^τ occurs implicitly on the right hand side of (2.4) the following modified scheme has actually been used.

The height values ϕ° are given initially. Then ξ° is computed from (2.5) and η° is obtained from (2.2).

We get

$$\xi_a^{1/2} = \xi^\circ + \frac{1}{2} \Delta t \cdot J(\eta^\circ, \phi^\circ)$$

where the subscript a denotes that the value is considered as a first approximation to the quantity in question. Then $\eta_a^{1/2}$ is computed from (2.2), and $\phi^{1/2}$ is obtained from $\xi_a^{1/2}$ through the solution of Poisson's difference equation (2.3). Here we utilize Liebmann's method of iteration taking ϕ° as the first approximation.

After this, the computation of the field at $\tau = 1, 2, 3, \ldots$ can proceed step by step in the following manner. Compute

$$\xi_a^\tau = \xi^{\tau-1} + \Delta t \cdot J(\eta_a^{\tau-1/2}, \phi^{\tau-1/2})$$

$$\xi^{\tau-1/2} = \frac{1}{2} (\xi_a^\tau + \xi^{\tau-1})$$

Then $\eta^{\tau-1/2}$ is obtained from (2.2), and the final value of ξ^τ is computed from

$$\xi^\tau = \xi^{\tau-1} + \Delta t \cdot J(\eta^{\tau-1/2}, \phi^{\tau-1/2})$$

For the use in the next step we also extrapolate

$$\xi_a^{\tau+1/2} = 2\xi^\tau - \xi^{\tau-1/2}$$

and compute the corresponding $\eta_a^{\tau+1/2}$. Then ϕ^τ is solved from (2.3), using the linear extrapolation $2\phi^{\tau-1/2} - \phi^{\tau-1}$ as a first approximation. Finally we extrapolate

$$\phi^{\tau+1/2} = \phi^\tau + \frac{1}{2} (\phi^\tau - \phi^{\tau-1})$$

and are ready for the next time step.

In the Liebmann process for solving (2.3) an over-relaxation factor of 1.2 was used. The solution was accepted when $| 1/4 (\Delta s)^2 \xi^\tau - (\bar{\phi}^\tau - \phi^\tau) |$ nowhere exceeded 0.3 m. As an average four or five iterations were needed. It can be shown that the Courant-Friedrich-Lewy criterion for computational stability is changed somewhat for a procedure as outlined above and the largest permissble value of Δt increases with 40—50 %. In most of the forecasts made here a time step of 1 ½ hour was used. The total time for a 24 hour forecast using 1,240 grid points was around 15 minutes.

It is quite obvious from these computations that 72 hours or somewhat more represents the limit with the method used. In most cases quite large oscillations in the vorticity field appear at this time, as can also be noticed in the height field. This is probably not due to computational instability in the sense of Courant-Friedrich-Lewy. A few forecasts were made using different time steps and no appreciable difference was obtained if the modified stability criterion mentioned above was satisfied. Instead these small scale disturbances are probably gradually activated through non-linear interactions, which is a normal feature of any such nonlinear process. The behaviour of such short waves is very badly represented by the finite difference approximations used. It therefore seems necessary to introduce systematic smoothing of the height field (or vorticity field) when extending the forecasts beyond about 48 hours. Such a procedure will be tried during the testing now in progess.

3. Forecasting with the barotropic model over 24, 48 and 72 hours

The results of the forecasts are summarized in table 1, r is the correlation coefficient between observed and computed changes, σ_x and σ_y are the root mean squares of the observed and computed changes, and ε is the mean error. The verification as presented here

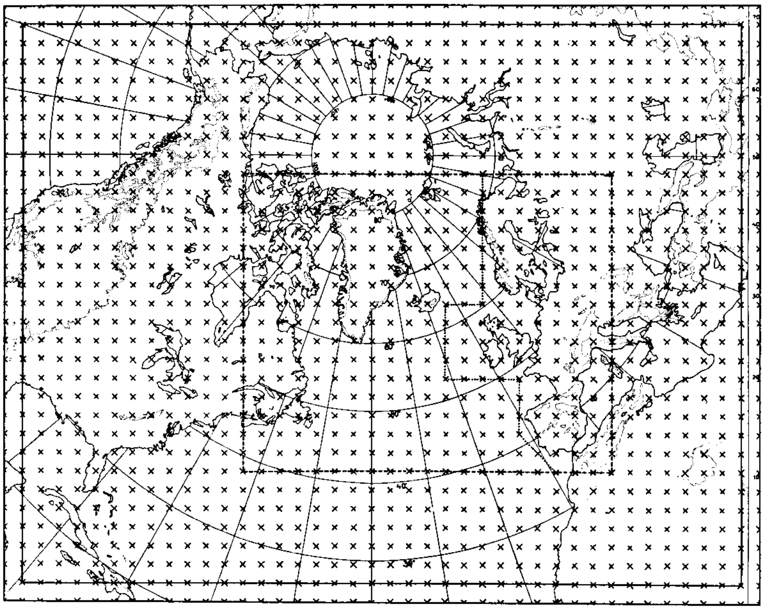

Fig. 1. Location of the grid used in the forecasts. The computations were made over the area limited by the inner solid line. At the initial time data was supplied also at the points outside to get an accurate value of the vorticity on the boundary. The verification was done over the area inside the dashed line and also over the land area indicated by the dotted line in the eastern part of this area.

was made over a comparatively small area over western Europe as indicated by the dotted line in fig. 1. The corresponding values have also been computed for a larger area indicated by the dashed lines in fig. 1. The average values of this latter comparison are given at the bottom of the table. The statistical quantities given in table 1 do not give a complete picture of the success or failure of the forecasts, but will still be useful for this discussion. To judge a forecast completely a careful inspection of the change maps in relation to the flow pattern is necessary. One such example will be given in the next section.

Table 1

Date	24^h barotropic forecast					48^h barotropic forecast					72^h barotropic forecast					48^h forecast; conventional methods				
	r	σ_x	σ_y	ε	ε/σ_x	r	σ_x	σ_y	ε	ε/σ_x	r	σ_x	σ_y	ε	ε/σ_x	r	σ_x	σ_y	ε	ε/σ_x
24/11-51 03	0.86	113	94	60	0.53	0.91	187	174	78	0.42	—	—	—	—	—	—	—	—	—	—
25/11-51 15	0.94	131	123	46	0.35	0.90	174	186	78	0.45	0.68	115	161	127	1.10	—	—	—	—	—
27/11-51 03	0.87	117	124	60	0.52	0.77	140	183	109	0.78	—	—	—	—	—	—	—	—	—	—
1/1 -54 03	0.78	68	60	53	0.79	0.63	121	118	105	0.87	—	—	—	—	—	—	—	—	—	—
21/9 -54 03	0.86	65	47	34	0.52	0.84	89	83	48	0.58	0.63	90	95	82	0.91	—	—	—	—	—
25/9 -54 03	0.91	87	81	40	0.46	0.84	110	121	84	0.76	—	—	—	—	—	—	—	—	—	—
26/9 -54 03	0.81	61	57	36	0.60	0.78	78	93	67	0.86	0.71	93	128	96	1.03	—	—	—	—	—
27/9 -54 03	0.60	46	37	40	0.85	0.74	81	64	58	0.72	0.63	89	92	87	0.98	0.70	81	56	84	1.04
28/9 -54 03	0.76	67	67	47	0.70	0.74	81	83	60	0.74	0.58	76	49	62	0.82	0.74	81	58	59	0.73
29/9 -54 03	0.93	71	71	28	0.39	0.84	94	85	55	0.59	0.62	136	115	113	0.83	0.80	94	46	65	0.69
30/9 -54 03	0.89	78	65	37	0.48	0.86	133	103	66	0.49	0.72	169	142	114	0.67	0.57	133	147	125	0.94
1/10-54 03	0.92	91	55	40	0.51	0.88	140	129	65	0.46	0.85	167	148	81	0.49	0.77	140	116	124	0.89
2/10-54 03	0.92	81	87	33	0.41	0.92	143	121	62	0.43	0.87	146	145	85	0.58	0.83	143	175	98	0.68
Mean	0.85	83	74	43	0.52	0.82	121	119	72	0.59	0.70	120	119	95	0.79	0.74	112	100	92	0.82
Verification over larger area	0.75	85	74	56	0.66	0.71	116	119	91	0.78	0.58	121	134	123	1.02					

The following points should be mentioned specifically:

1) On the whole the agreement between observed and computed changes was worse over the larger area than the one only covering parts of the European continent and adjacent ocean areas. In the western parts the influences from the boundaries may have been of some importance after 48 and 72 hours, but already in the 24-hour forecasts a significant difference shows up. Undoubtedly this is to some extent the result of the comparatively few radiosonde observations over the ocean. However, another effect is also important. Computing the average errors in the 48-hour forecast of the height changes we find that they had a maximum in the vicinity of New Foundland and Nova Scotia (average of 13 forecasts). The computed changes were here on an average 80—100 m too high. This means that the absolute vorticity in reality was larger in this region than was computed. Since the barotropic model assumes conservation of the absolute vorticity this result indicates that this region in reality is a source region of vorticity. This result is not surprising. From experience we know that most cyclones are formed along the American sea border where the baroclinicity of the atmosphere on an average is large. Furthermore the thermal contrast between the continents and the ocean is larger at the American east coast than at the European west coast. *The results obtained indicate that baroclinic processes and non-adiabatic influences from the surface of the earth are less important over the eastern Atlantic and western Europe than over the American east coast and western Atlantic for forecasting changes of the 500 mb flow*. Of course this is not necessarily true in individual weather situations, but only expresses an average condition.

2) The 48-hour forecasts compare well with the 24-hour forecasts and also those for 72 hours were in most cases of definite value. It is interesting to notice that in some cases the correlation coefficient was larger and the relative error ε/σ_x smaller for the 48 hour forecasts than for those over 24 hours. On September 27 this was partly a result of a deepening cyclone over the Baltic during the first 24-hour period, which moved out of the area during the following day. However, on November 24, 1951 it was not the result of such a development. Instead we see that σ_x is considerably greater for 48 hours than for 24 hours (as in all cases presented here). This means that the quasi-periodic changes caused by the large scale patterns in the atmosphere on an average have a half period which is larger than one day. In the changes over 24 hours the large scale changes thus play a relatively less important role than those with shorter period since they have not yet reached their maximum intensity. However, the barotropic model is less applicable to systems of smaller scale and furthermore the finite difference approximations become worse, which explains this initial increase of the correlation coefficient. A series of 12-hour forecasts would probably show consistently lower correlation coefficients than those here obtained for 24 hours.

3) It is to be expected that the small scale atmospheric systems are poorly described by the barotropic model and the finite difference method also introduces large errors in computing the changes of such systems. This is clearly seen from an inspection of the vorticity field. The amplitude of the vorticity is considerably larger in she small systems than in the larger ones, while still the latter are most important for determining the general flow. Even in the very best forecasts there are large differences between the observed and forecast vorticity fields. This fact speaks in favour of the introduction of a systematic smoothing in the course of the computations.

4) The notion that the barotropic model describes large scale process in the atmosphere better than those of smaller scale seems not to be valid for the very largest systems. This is indicated by the fact that the correlation between computed and observed *mean* changes over the verification area in these 13 forecasts is somewhat smaller than the correlation of the changes at different points within the area.

5) The results presented here for 24 hours are on an average better than those given in the previous report II. Four synoptic situations have been treated both with the smaller area of initial data used in II and the larger one used here. For the three November cases in 1951 the differences in the results are hardly significant. Those made with the larger area seem to be slightly better. The forecast from January 1, 1954 was previously a failure with a correlation of 0.17 and a mean error of 169 m. It

was quite obvious that the boundary assumptions to a large extent were responsible for this result. The computations using the larger area resulted in an improvement to the values $r = 0.78$, $\varepsilon = 53$ m. The verification was not done over quite the same area but the values are still comparable.

6) A study of the error maps shows that this field to some extent is associated with the existing flow pattern. A knowledge of the error field over one 24-hour period permits us to draw some conclusions concerning the errors during the following 24-hour period. In whatever way these errors arise it should be possible to utilize the information they contain for improving the following forecast.

7) The last eight forecasts (September 25—October 2) were made in cooperation with the Weather Service of the Swedish Air Force and used in their daily routine forecasting. During the same period 48-hour forecasts were made by the Swedish Weather Bureau with conventional extrapolation methods similar to those developed by SCHERHAG (1948). The results of those are also given in table 1. It should be remarked that no major baroclinic development occurred during this period which at least to some extent might have been caught by the conventional methods, but certainly not by the barotropic model. It is of course also conceivable that some other method of extrapolation may give better results than the one used here. For 24-hours the conventional methods seem to have given somewhat better results than the barotropic model (not shown in the table). However, for 48 hours the barotropic model definitely was superior in these cases.

8) Very large errors sometimes appear after about 48 hours. They are among other things characterized by the fact that the subtropic anticyclones are intensifying considerably as well as the polar cyclones resulting in a general intensification of the gradients. In a few cases absurd values are obtained for the geopotential field as for example the value of 6,160 m in the centre of the anticyclone over the Eastern United States in the 72 hour forecast from October 2 (fig. 5). In the same forecast one also finds quite a strong rise over most part of the large verification area towards the end of the forecast period. Thus the computed mean rise over this area from 24 to 72 hours was more than 100 m, a value never observed in reality over such large an area.

It is quite clear that these large scale errors hardly can be the result of baroclinic developments in the real atmosphere. On the other hand the non-barotropic processes associated with the influences from the surface of the earth seem to be of some importance in forecasts over 48 hours or more. However, correcting for this in a similar way as proposed by CLAPP (1953) still leaves large errors unexplained.

Again we come back to the boundary assumptions. Keeping the vorticity at a fixed value on the boundary may cause large errors to build up gradually. Let us for example assume that a comparatively low value of the absolute vorticity is assigned to a portion of a southern boundary across which inflow takes place. This value is then kept fixed in the course of the following computation. The flow north across the boundary turns anticyclonically to the right (northern hemisphere). Here gradually an anticyclone will form meaning an intensification of the southerly flow to the west. Thus, a day later the vorticity carried in across the boundary is brought further to the north than on the previous day and also the anticyclone extends further to the north. At these higher latitudes the same small absolute vorticity coming from the boundary means a still more intense anticyclonic circulation because of the variation of the Coriolis parameter. In view of such effects *it may in many cases be necessary to place the boundaries in such a way that no flow takes place across them, if the forecasts are to be extended beyond 48 hours or else modify the assumptions on the boundaries to exclude developments of this kind.*

It still does not seem likely that all major errors in the forecasts will be removed by corrections of this type. We notice for example in the 48 hour forecast from October 2, 1954 (fig. 2) an anticyclogenesis over the sea between Greenland and northern Scandinavia, which did not occur in reality. This may be the result of the fact that the cyclonic vorticity in the cyclone to the east of the British Isles was over-estimated by the geostrophic approximation, which caused a turning of the flow more sharply than actually occurred for example after 48 hours (fig. 4). This error in the evaluation of the vorticity essentially

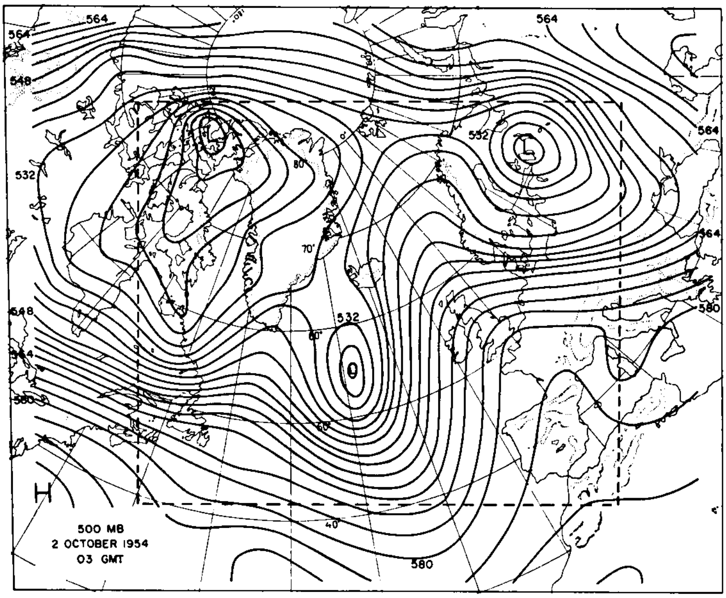

Fig. 2. 500 mb contours on October 2, 0300 GMT 1954. The heights are given in decameters.

depends upon the curvature of the flow and is small in the case of straight flow. Therefore, with the assumption of conservation of the absolute vorticity the *geostrophic* vorticity is not conserved for a particle at one time being in a region of strong curvature at another time in more or less straight flow.

The series of questions raised above makes it clear that a reanalysis of the barotropic model is highly desirable. We shall return to this problem in section 5.

Forecasts with Fjørtofts graphical method

by O. Haug

For the period 25 September to 2 October 48 hour forecasts have been made by means of Fjørtofts method in order to compare the results thus obtained with those obtained numerically with the aid of BESK. For details of the method we refer to the paper by Fjørtoft (1952). The computations were here done using a modified formula, which also takes into consideration the changes in the mean field, while the mean of the mean field is kept fixed in time. This method will be discussed by Fjørtoft in a forth-coming paper. The gridsize used for constructing the mean map was 600 km at 50° N and for constructing the mean of the mean 1,200 km.

The prognoses from the first four days were about as successful as those made with the machine. The correlation coefficients for the observed and computed changes evaluated over the large area indicated in fig. 1 are given in the following table.

Table 2

	25—27 sept.	26—28 sept.	27—29 sept.	28—30 sept.
BESK	0.58	0.77	0.64	0.83
Fj. method ..	0.70	0.60	0.60	0.71

During the later period there was a very pronounced increase of the amplitude of the large scale wave pattern over the Eastern Atlantic and Western Europe. The forecasts by BESK became better (see table 1) and apparently the development was to a considerable extent barotropic. The forecasts with the graphical method, on the other hand, were not as good, which in particular was true regarding the forecast of the amplitude of the waves.

The main source of errors in the forecasts by Fjørtoft's method seem to be the actual variations of the mean of the mean field, which is supposed to be constant during the forecast interval of 48 hours. During the first part of the period only small changes in the mean of the mean field occurred, while it changed considerably during the later part. There are, however, reasons to believe that these pronounced changes were quite extreme. Such changes do not occur very often, but they usually represent major changes of the weather in the area concerned and are extremely important for the weather forecasting.

The displacement of troughs and ridges were in most cases fairly well forecast by Fjørtoft's method. The error in displacement of the trough and ridge, moving from the Atlantic in towards Europe during the later part of the period are given in the following table 3 as the percentage of the time displacement. The figures represent the mean of the errors of displacement at the latitude circles 60° and 50° N.

Table 3

Percentage error in prognosis of displacement of trough.

	29 sept. — 1 oct.	30 sept. — 2 oct.	1—3 oct.	2—4 oct.
BESK	9 %	20 %	7 %	26 %
Fj. method ..	23 %	77 %	18 %	33 %
Percentage error in prognosis of displacement of ridge				
BESK	27 %	21 %	7 %	10 %
Fj. method ..	10 %	18 %	17 %	4 %

It may be of some interest to see that the trough displacement was considerably better forecast by BESK, while the motion of the ridge was very well forecast using graphical methods, even better than with the machine.

It should finally be added that it takes 4—5 hours for one man to prepare a 48-hours forecast with Fjørtoft's method. The time can be reduced considerably if two persons work simultaneously.

4. Barotropic forecasts from October 2, 1954, 03 GMT

A correlation coefficient or mean error certainly gives a very poor description of the success or failure of a forecast. We shall therefore in some detail discuss some of the forecasts made in this series and have chosen those based on the map of October 2, 1954, 03 GMT. It was the most successful forecast over north-western Europe for 48 and 72 hours and it is interesting in view of the prevailing synoptic situation. On the other hand the results for the western parts of the verification area (western Atlantic) were worse than on the average (fig. 2—5).

The initial map was characterized by two well defined troughs, one along 30° E, the other one at 25—30° W. The ridge between these troughs had been intensifying during the last 24 hours. Further west another trough was situated on the lee side of the Rocky Mountains. Off the American west coast a very intense blocking ridge had remained stationary for some days (not shown on the map). During the following 48 hours the ridge over the Bristish Isles intensified and moved east and the trough further west also advanced quite rapidly eastwards. As is seen from fig. 4 the 48 hour forecast of the positions of the trough and ridge was almost perfect. The very characteristic changes in the shape of the ridge in that the distance between the two troughs on both sides became smaller is clearly indicated in the forecast. The westerly current from the Atlantic pushing westward towards central Europe had reached the zero meridian on October 4 in the morning. In the forecast it had advanced somewhat further and a westerly current was gradually being generated over central Europe. During the following 24-hour period this development continued and the weather was radically changed in that humid maritime air replaced the dry polar air that had earlier been brought in from the north over most parts of central Europe. It is very interesting indeed that such a development of the decline of the ridge and the intensification of the anticyclone over northern Finland and Russia in principle could be forecast three days in advance. It is also interesting to notice that the correlation between the observed changes between October 4 and 5 and those computed for the same 24-hour period using the initial map two days earlier was 0.65.

In other regions the errors in the forecasts were larger. Over the Norwegian Sea the current never turned back towards NNW as was forecast. As mentioned in the previous section this possibly may be explained as a result of an overestimation of the cyclonic vorticity in the trough at 30° W on October 2. The depth of the centre may also have been

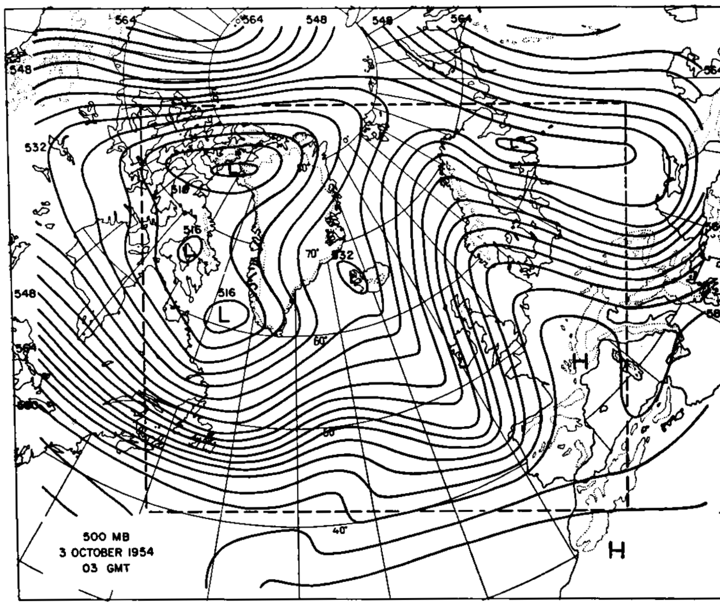

Fig. 3 a. 500 mb contours on October 3, 0300 GMT, 1954.

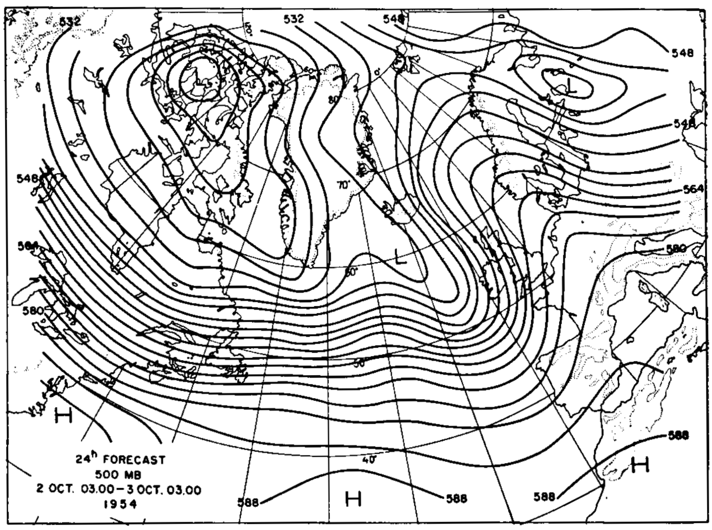

Fig. 3 b. 24-hour forecast of 500 mb contour field from October 2, 0300 GMT to October 3, 0300, 1954.

erroneously analyzed as no observations existed close to it.

To get an idea of the importance of the geographically fixed errors the following computation was carried through. The 48-hour forecast was corrected with the aid of the mean 48-hour errors found from the other 12 forecasts given in table 1. In this way the correlation coefficient over the smaller verification area was changed from 0.92 to 0.93 and the mean error from 62 to 59 m. The improvement is hardly significant. Probably a more pronounced improvement in general would be obtained at the east coast of America. Furthermore it would of course be better to introduce corrections of this type in the course

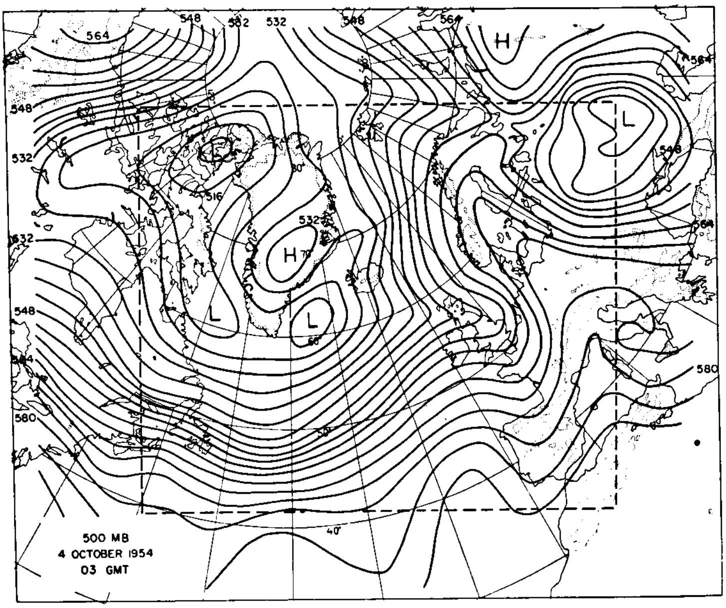

Fig. 4 a. 500 mb contours on October 4, 0300 GMT, 1954.

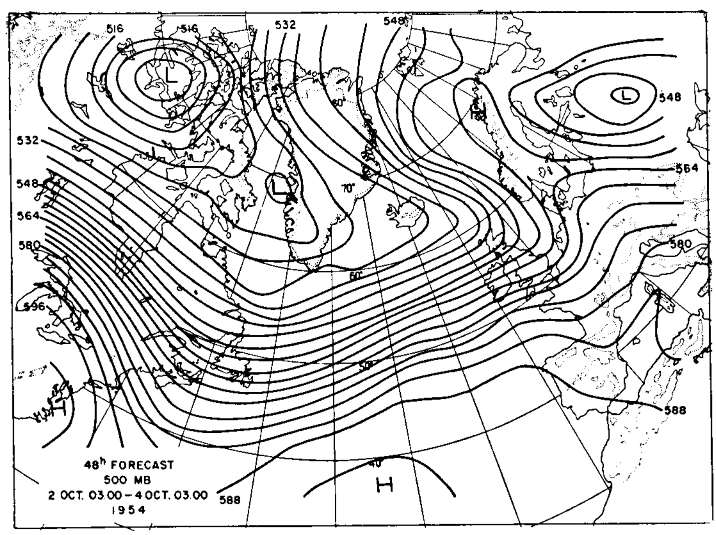

Fig. 4 b. 48-hour forecast of 500 mb contour field from October 2, 0300 GMT to October 4, 0300 GMT, 1954.

of the computations, since such errors cause new errors if not continuously removed.

5. The barotropic model

There are a number of reasons, why the barotropic, two-dimensional model was chosen for the first attempts in numerical forecasting. The quasi-two-dimensional character of the motion of the atmosphere had been stressed repeatedly by Rossby and his collaborators (e. g. ROSSBY 1939). The transformation of potential energy into kinetic energy only amounts to some 10—20 % of the total kinetic energy of the atmosphere per day and to a first approximation the changes of atmospheric flow-pattern represent a redistribution of

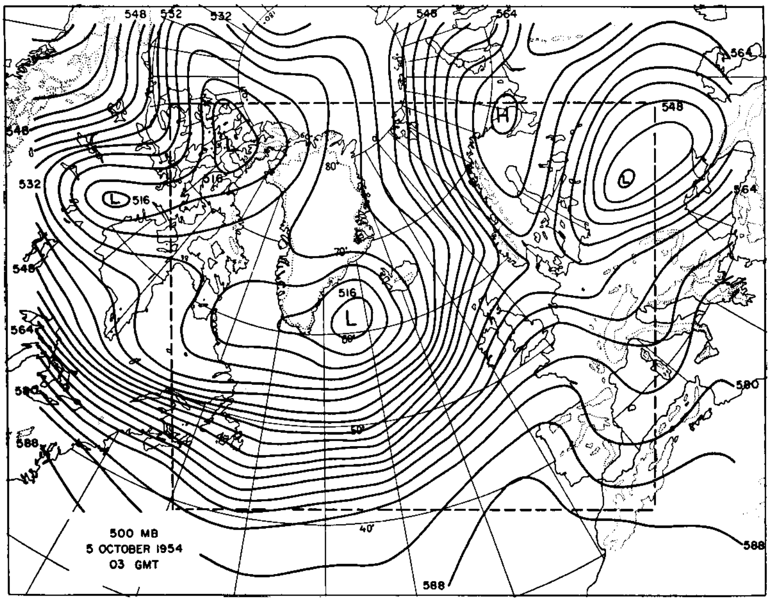

Fig. 5 a. 500 mb contours on October 5, 0300 GMT, 1954.

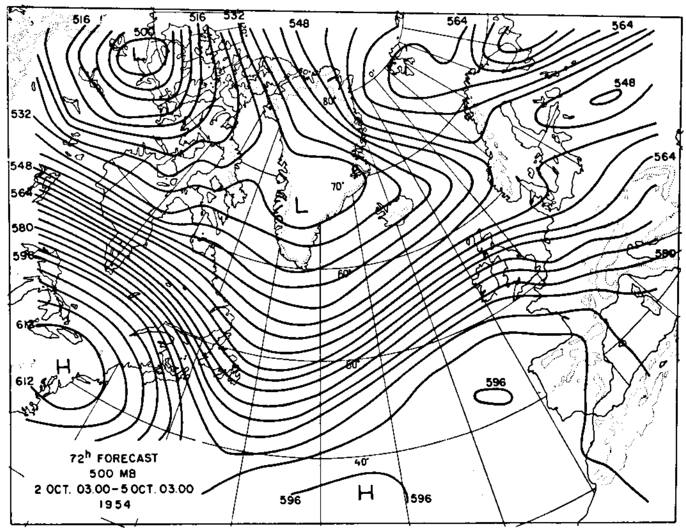

Fig. 5 b. 72-hour forecast of 500 mb contour field from October 2, 0300 GMT to October 5, 0300 GMT, 1954.

kinetic energy. Above all it is important to start from the simplest possible idea about the dynamics of the atmosphere and gradually proceed to more complicated models. In doing so we can get a better understanding of the relative importance of various processes in the atmosphere. In that sense the barotropic model offers an excellent starting point.

There are several advantages in using an internally consistent model of the atmosphere in such an approach. It is easier to visualize the behaviour of the field of motion and to clarify the importance of various *physical* factors. In starting from the general equations in three dimensions and attempting to derive a set of forecast-equations by successive approximations inconsistencies are easily introduced, the importance of which is difficult to determine.

For this reason one has chosen a homogeneous and incompressible fluid of a finite depth. We shall study it both assuming a fixed upper surface of this fluid (i.e. put div $\mathbf{v} = 0$) as well as a free surface. The following three equations describe the motion of the fluid with a free surface.

$$\left.\begin{aligned} &\frac{du}{dt} - fv = -g\frac{\partial D}{\partial x} \\ &\frac{dv}{dt} + fu = -g\frac{\partial D}{\partial y} \\ &\frac{dD}{dt} + D\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \end{aligned}\right\} \qquad (5.1)$$

where u and v are the components along the x- and y-axes respectively of the horizontal wind $\mathbf{v}$, f is the Coriolis parameter, D denotes the depth of the fluid and d/dt only contains the horizontal convective accelerations. We have here made use of the hydrostatic equation. From these equations the well-known vorticity equation is derived

$$\frac{d}{dt}(\zeta + f) = -(f + \zeta)\,\mathrm{div}\,\mathbf{v} \qquad (5.2)$$

where ζ denotes the relative vorticity. In case of a fixed top on the fluid the right side is equal to zero and this simplified form of (5.2) is the basis of all barotropic forecasts published up to the present time.

Any results derived for such a model must be "translated" in terms of the real atmosphere. Thus CHARNEY (1949) has introduced the concept of the equivalent barotropic level of the atmosphere and justified the use of data at 500 mb in forecasting with (5.2). In (a) we shall give some further considerations on this problem starting from the general three-dimensional equations.

The geostrophic approximation has been of great importance for our possibilities of utilizing eq. (5.2) for forecasting purposes. In an other article in this issue of Tellus Charney discusses the more general equation relating wind and pressure in the nondivergent case. (CHARNEY 1955). Similar considerations have been made by the author and a detailed discussion of this generalization of the geostrophic relation will be given in (b).

In section (c) we shall discuss the importance of removing the restriction of zero divergence. In doing so the balance equation discussed in (*b*) becomes more complicated and some modifications of it are presented in (*d*).

It is hoped that this detailed discussion of the barotropic model shall serve the two-fold purpose of improving the present procedure in applying the barotropic model and form a starting-point for a discussion of more complicated models of the atmosphere.

a. *The equivalent barotropic atmosphere*

There exist marked differences between the real atmosphere and the barotropic nondivergent model. Thus the wind increases from relatively small values at the surface of the earth to a maximum at the tropopause level. The vertical velocity in reality seems to have a maximum somewhere in the middle of the atmosphere and is equal to zero at the surface of the earth and also in the vicinity of the tropopause, implying opposite sign of the divergence in the lower and upper part of the troposphere. A number of approximations therefore are necessary to arrive at the simplified equation used in the present computations with the barotropic model. The question arises if some of these simplifications can be removed without having to introduce several parameters. It is clear that some kind of multiple parameter model ultimately will be used in numerical forecasting but we shall here see if it is possible to make any appreciable improvements within the framework of the simple barotropic model. Let us for this purpose turn back to the complete hydrodynamic equations in three dimensions.

For this discussion we shall make use of the hydrostatic relation

$$\frac{\partial p}{\partial z} = -g\varrho \qquad (5.3)$$

with the aid of which we introduce p as our vertical coordinate. The two horizontal components of the equation of motion are combined into the vorticity equation and the divergence equation. At this stage we shall only make use of the first one. The divergence equation will be used in 5 b and 5 d to relate the pressure and wind fields to each other. The complete vorticity equation reads

$$\frac{\partial \zeta}{\partial t} + \mathbf{v} \cdot \nabla (\zeta + f) = -\omega \frac{\partial \zeta}{\partial p} + (f + \zeta) \frac{\partial \omega}{\partial p} + $$
$$+ \frac{\partial \omega}{\partial y} \cdot \frac{\partial u}{\partial p} - \frac{\partial \omega}{\partial x} \cdot \frac{\partial v}{\partial p} \qquad (5.4)$$

where $\omega = dp/dt$ and where we have made use of the continuity equation. At this stage CHARNEY (1949) makes the following assumptions.

1) The variation of the wind with height is fairly well represented by

$$\mathbf{v}(x, y, p) = A(p)\bar{\mathbf{v}}(x, y) \qquad (5.5)$$

where $A(p)$ is an empirical function for which

$$\frac{1}{p_0} \int_0^{p_0} A(p)\, dp = 1 \qquad (5.6)$$

$\bar{\mathbf{v}}$ is the mean velocity averaged over the vertical.

2) $\zeta \ll f$ and therefore the vertical advection of vorticity, $(\omega \cdot \partial\zeta/\partial p)$, and $\zeta \cdot \partial\omega/\partial p$ are small terms in comparison with $f \cdot \partial\omega/\partial p$.

3) The turning of the vortex tubes expressed by $(\partial\omega/\partial y \cdot \partial u/\partial p - \partial\omega/\partial x \cdot \partial v/\partial p)$ can be neglected.

By integrating over the vertical Charney shows that

$$\frac{\partial \zeta^\star}{\partial t} + \mathbf{v}^\star \cdot \nabla (\zeta^\star + f) = 0 \qquad (5.7)$$

Here $\zeta^\star$ and $\mathbf{v}^\star$ represent the vorticity and velocity at a level $p^\star$ where

$$A(p^\star) = \frac{1}{p_0} \int_0^{p_0} [A(p)]^2 dp \qquad (5.8)$$

It turns out that $p^\star$ is close to 500 mb on an average. This is the justification for using the simple vorticity equation valid for an incompressible and homogeneous atmosphere for forecasting the changes at 500 mb in the real atmoshpere.

However, none of these three assumptions are valid in reality, as has been pointed out several times by other authors. The use of multiple-parameter models represent attempts to remove the restrictions introduced by these assumptions in particular the first one. We shall here only discuss the barotropic model and therefore the first assumption will be retained in spite of the fact that large errors are introduced in highly baroclinic fields.

The second assumption introduces errors, which can easily be estimated in an atmosphere with the vertical distribution of the wind given by (5.5). Consider for example a region with a positive value of ζ. Because of the general character of $A(p)$ it follows that $\partial\zeta/\partial p < 0$. Hence, in case this is a region of upward motion $\omega \cdot \partial\zeta/\partial p$ averaged through the whole troposphere is a positive quantity. In such an area of upward motion the divergence $(= -\partial\omega/\partial p)$ is positive at upper levels and negative at lower levels, but the integrated value over the vertical is close to zero. However, if ζ is positive, it is larger at higher levels than close to the surface of the earth $(\partial\zeta/\partial p < 0)$ and the average of $\zeta \cdot \partial\omega/\partial p$ is therefore also different from zero. It is easily seen that these two terms give a contribution in the same direction. We shall here give a rough estimate of the importance of this effect. As will be seen from the following derivation the result depends on the assumption of the vertical distribution of the horizontal wind and the vertical velocity. It is difficult to see what the various assumptions will mean and only numerical forecasts using the final formula will answer this question.

For simplicity we shall assume

$$A(p) = 2 \cdot \frac{p_0 - p}{p_0 - p_1} \qquad (5.9)$$

which gives $\mathbf{v}(x, y, p_0) = 0$ and $\mathbf{v}(x, y, p_1) = 2\bar{\mathbf{v}}$. p_0 is the pressure at the surface of the earth and p_1 the pressure at the tropopause. Let us restrict the following considerations to the troposphere. (5.9) then describes average conditions quite well (BOLIN, 1953a).

It has already been mentioned that the vertical velocity seems to have an extremum somewhere in the middle of the troposphere. This is mainly a result of the existing vertical distribution of the wind. If it is accepted that the vertical variation of the wind can be described fairly well with the expression (5.9) the atmosphere must contain processes that try to maintain such a distribution. This means that $\partial\zeta/\partial t$ in (5.4) also must be a function of p approximately represented by $A(p)$. Assuming for a moment that $\zeta \ll f$ and (5.4) can be written

$$\frac{\partial \zeta}{\partial t} + \mathbf{v} \cdot \nabla (\zeta + f) = f \cdot \frac{\partial \omega}{\partial p} \qquad (5.10)$$

It is then easy to show that $|\omega|$ must have a maximum in the middle of the troposphere to have this equation satisfied at all levels between $p = p_0$ and $p = p_1$. For the case that $\omega(p_0) = \omega(p_1) = 0$ one obtains

$$\omega = \frac{\bar{\mathbf{v}} \cdot \nabla \bar{\zeta}}{3f} \Delta p(\pi^3 - \pi^2 - \pi + 1) \qquad (5.11)$$

where

$$\pi = \frac{2p - p_1 - p_0}{p_0 - p_1}; + 1 \leq \pi \leq - 1 \qquad (5.12)$$

and $\Delta p = \frac{1}{2}(p_0 - p_1)$. In view of the fact that $f \cdot \partial\omega/\partial p$ is the most important term containing ω in (5.4) one may draw the conclusion: ω has an extremum in the vicinity of the mid-troposphere which is the result of the fact that the wind increases approximately linearly with height. This forms the basis for the assumption of a parabolic variation of the vertical velocity with height implicit in most two-parameter models (cf. EADY, 1952; ELIASSEN 1952).

It is quite obvious that any assumption of the form (5.5) means an inconsistency because the equations governing the motion of the atmosphere are non-linear in their character, which means a continuous activation of higher modes also along the vertical. Instead of requiring that the vorticity equation is satisfied at all levels we shall therefore follow the procedure used by EADY (1952) and ask for the vorticity equation to be satisfied if integrated over the vertical in various ways. Let us for simplicity assume a parabolic distribution of ω with p instead of (5.11), since the detailed structure is of no great importance in estimating the other terms depending upon ω in the vorticity equation.

$$\omega = 4\omega_m \frac{(p - p_1)(p_0 - p)}{p_0 - p_1} \qquad (5.13)$$

It is here also assumed that $\omega(p_0) = \omega(p_1) = 0$, an assumption which will be discussed later (5 c). Introducing (5.9) and (5.13) into (5.4) and integrating between p_0 and p_1, we obtain

$$\frac{\partial \bar{\zeta}}{\partial t} + \frac{4}{3} \bar{\bar{\mathbf{v}}} \cdot \nabla \bar{\zeta} + \bar{\mathbf{v}} \cdot \nabla f = \frac{4}{3} \frac{\omega_m \bar{\zeta}}{\Delta p} \qquad (5.14)$$

The turning of the vortex tubes has been neglected for the time being. Another relation is obtained in the same way as by EADY (l.c.). Instead of adding the contribution from each half of the atmosphere we subtract them. Thus

$$\frac{\partial \bar{\zeta}}{\partial t} + 2\bar{\mathbf{v}} \cdot \nabla \bar{\zeta} + \bar{\mathbf{v}} \cdot \nabla f = 2 \frac{\omega_m}{\Delta p} (f + \bar{\zeta}) \qquad (5.15)$$

From (5.14) and (5.15) we derive the following expression for the vertical velocity

$$\omega_m = \frac{\Delta p}{3f + \bar{\zeta}} \bar{\mathbf{v}} \cdot \nabla \bar{\zeta} \qquad (5.16)$$

and the vorticity equation may be written

$$\frac{\partial \bar{\zeta}}{\partial t} + \frac{4f}{3f + \bar{\zeta}} \bar{\bar{\mathbf{v}}} \cdot \nabla \bar{\zeta} + \bar{\mathbf{v}} \cdot \nabla f = 0 \qquad (5.17)$$

To be able to compare this with (5.7) we introduce $\zeta^\star = {}^4/_3 \cdot \bar{\zeta}$ and $\mathbf{v}^\star = {}^4/_3 \cdot \mathbf{v}$. Thus

$$\frac{\partial \zeta^\star}{\partial t} + \frac{4f}{4f + \zeta^\star} \mathbf{v}^\star \cdot \nabla \zeta^\star + \mathbf{v}^\star \cdot \nabla f = 0 \qquad (5.18)$$

If $\zeta^\star \approx f$ an error of about 25 % is made in evaluating the vorticity change by using equation (5.7) instead of (5.18). The approximation made in disregarding the horizontal gradients of the vertical velocity is in most cases permissible. These terms are of course of principal importance in cases of shear lines or intense jet streams, but it is doubtful if such phenomena can be treated with a one-parameter model, which describes the vertical wind shear relatively crudely.

Equation (5.18) is interesting in that there is a systematic difference between areas of cyclonic and anticyclonic relative vorticity. The former move more slowly and the latter more rapidly than the pure advection indicates. Obviously the absolute vorticity is not conserved. However, the changes depend upon the gradient of the relative vorticity, which is small $(= - \nabla f)$ in areas where $\eta^\star$ has a maximum or a minimum. Thus the total range within which $\eta^\star$ varies over the field remains approximately constant in time.

It is difficult to judge from the error maps of the series of computations presented above, if (5.18) would mean an improvement of the present method. This will be tested in coming forecasts. It should be kept in mind, however, that it is based on the assumption that the vertical variation of wind as well as vertical velocity is the same everywhere. We know, for example, that there are systematic differences between cyclones and anticyclones noticeable for example in systematic variations in the height of the equivalent barotropic level (cf. BOLIN and CHARNEY, 1951). This may counteract the effects described.

b. *The relation between wind and pressure in the nondivergent case*

All three forecast-equations (5.2), (5.7) and (5.18) have the characteristic feature that a knowledge of the wind field at one instant permits an evaluation of the development in time without any knowledge of the pressure field. In reality the pressure field is better known than the wind field and therefore the wind observations usually are supplemented by the geostrophic wind computed from the pressure field. *The reason for introducing the geostrophic approximation in the barotropic model is therefore merely to be able to use both pressure and wind observations as initial data in the computations.* It is not necessary from the computational point of view. We shall here discuss the more general relation between wind and pressure in some detail.

Let us for this purpose consider an internally consistent model of the atmosphere. At this first stage we again choose a homogeneous and incompressible fluid with a fixed upper surface. Thus div $\mathbf{v} \equiv 0$. Taking the divergence of the equations of motion we arrive at the following equation relating the pressure field and the wind field to each other:

$$\nabla^2\phi = f\zeta + 2\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}\right) - f_y u + f_x v \tag{5.19}$$

where $f_y = \partial f/\partial y$, $f_x = \partial f/\partial x$ and $\phi = p/\varrho$. Since the motion is assumed to be non-divergent we can introduce a stream function ψ defined by

$$\left.\begin{aligned} u &= -\frac{\partial\psi}{\partial y} = -\psi_y \\ v &= \frac{\partial\psi}{\partial x} = \psi_x \end{aligned}\right\} \tag{5.20}$$

Thus

$$\nabla^2\phi = f\nabla^2\psi + 2(\psi_{xx}\psi_{yy} - \psi_{xy}^2) + f_y\psi_y + f_x\psi_x \tag{5.21}$$

If neglecting the variation of the Coriolis parameter and the non-linear terms we obtain

$$\nabla^2\phi = f\zeta \tag{5.22}$$

which is the approximate form used in evaluating the vorticity with the aid of the geostrophic approximation. This relation is used in all barotropic (and barolinic) computations published so far. The neglections made in using (5.22) are of two different kinds. Integrating (5.21) over an area S and making use of Gauss theorem gives

$$\left.\begin{aligned} \int_L \frac{\partial\phi}{\partial n}\,dl = \iint_S f\cdot\zeta dS + \int_L (\mathbf{v}\cdot\nabla\mathbf{v})_n\,dl + \\ \iint_S (f_y\psi_y + f_x\psi_x)\,dS \end{aligned}\right\} \tag{5.23}$$

Here n means the normal direction to the boundary, l is the coordinate along the boundary L. From (5.21) and (5.23) we see that the non-linear term $(\psi_{xx}\psi_{yy} - \psi_{xy}^2)$ is of importance for a correct description of relatively small scale systems. Locally it may approach the size of the two terms retained in using the approximate relation (5.22). However, if integrating over a large area the positive and negative contributions to the integral approximately compensate each other. The term depending on the variation of the Coriolis parameter, on the other hand, is locally small, but has the same sign as the westerly component of the motion and may therefore have the same sign over large areas. It is of particular importance in describing the large scale systems in the atmosphere.

It is quite clear from (5.21) that ϕ is uniquely determined, if ψ is known over the area and ϕ is given on the boundaries. In reality, however, we observe the height of the pressure surfaces, i.e. the pressure field, and winds at these levels. Let us assume that the wind field at 500 mb is to be used in the non-divergent barotropic model. The problem is then to determine the stream function ψ initially so that the wind pattern thus defined and the corresponding pressure field defined by (5.21) agree with the observations as well as possible. The present way of analysing a 500 mb map is essentially an analysis of the pressure field, using the observed winds as a guide in areas with few observations. We shall at this moment assume that it represents ϕ and our problem is to determine ψ from a given pressure field.

Looking upon (5.21) in this way it is a special case of the Monge-Ampère's differential equation and possesses certain characteristic features which are of interest here. It is of the elliptic type if

$$\nabla^2\phi - f_y\psi_y - f_x\psi_x > -\frac{f^2}{2} \qquad (5.24)$$

It then has two and only two solutions, if the boundary values of ψ are specified. To explore the character of the non-linear terms in the equation (5.21) and this criterion we shall assume that $f_x = f_y = 0$ for the time being. Thus (5.24) becomes

$$\frac{\nabla^2\phi}{f} > -\frac{f}{2} \qquad (5.25)$$

This means that the geostrophic vorticity must never be smaller than $-f/2$. In this case (5.21) may be written

$$(2\psi_{xx}+f)(2\psi_{yy}+f) - 4\psi_{xy}^2 > 0 \quad (5.26)$$

Thus the product $(2\psi_{xx}+f)$ $(2\psi_{yy}+f)$ always is positive and either both factors are positive or both are negative. The two possible solutions are characterized by

$$\left.\begin{aligned} \zeta_1 = \nabla^2\psi_1 > -f \\ \zeta_2 = \nabla^2\psi_2 < -f \end{aligned}\right\} \qquad (5.27)$$

or

$$\left.\begin{aligned} \eta_1 > 0 \\ \eta_2 < 0 \end{aligned}\right\} \qquad (5.28)$$

η being the absolute vorticity. For continuity reasons it therefore follows that the solution of (5.21) is characterized by the fact that the absolute vorticity is positive everywhere or negative everywhere. In the northern hemisphere of course the only solution of interest is the one where $\eta > 0$. In the case of a circular vortex we here recognize a well-known fact. For a given pressure field there exist two possible solutions to the gradient wind equation.

If the inequality (5.25) is not fulfilled equation (5.21) is hyperbolic in this region. However, this is usually the case only in very limited areas. For such an area specifying ϕ in the interior of it and ψ on the boundaries usually means an over-determination of the problem and no solutions exist. The features of the equation in this sense are very similar to those of linear hyperbolic equations of second order.

It is not unusual to find areas on a 500-mb map within which (5.25) is not satisfied. If introducing the notations

$$\left.\begin{aligned} A &= \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = -2\psi_{xy} \\ B &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \psi_{xx} - \psi_{yy} \end{aligned}\right\} \qquad (5.29)$$

we can write (5.21)

$$\nabla^2\phi = f\zeta - \frac{1}{2}(A^2 + B^2 - \zeta^2) \qquad (5.30)$$

(cf. SHERMAN, 1952; PETTERSSEN, 1953). We here still have assumed that $f_x = f_y = 0$. A and B are the two components of the deformation field. Whatever values of ζ are introduced in (5.30), $f\zeta + \frac{1}{2}\zeta^2 \geqslant -f^2/2$. Therefore if (5.25) is not satisfied the deformation field is of importance (or it is not permissible to make the assumption of non-divergence). *It is not possible to draw the conclusion that the absolute vorticity is negative by only consulting the pressure field (and its derivatives) in the point under consideration.*

Let us next consider the terms depending upon the variation of the Coriolis parameter. To clarify some facts regarding their importance we shall for a moment neglect the non-linear terms and thus write (5.21) as

$$\frac{1}{f}\nabla^2\phi = \nabla^2\psi + \frac{f_y}{f}\psi_y \qquad (5.31)$$

where we for simplicity have assumed that f is a function of y only. In middle latitudes the flow usually has a westerly component, i. e. $\psi_y < 0$. The neglection of the term depending upon the variation of the Coriolis parameter then means an under-estimation of vorticity. Differentiating (5.31) with respect to time and making use of the vorticity equation

$$\frac{\partial\zeta}{\partial t} = \frac{\partial\zeta_g}{\partial t} + f_y\frac{\partial u}{\partial t} = -\mathbf{v}\cdot\nabla(\zeta + f) \quad (5.32)$$

ζ_g being the geostrophic vorticity $\zeta_g = f^{-1}\nabla^2\phi$. (5.32) shows that the geostrophic vorticity is not conserved but decreases in an area were the westwind increases and vice versa. Accordingly errors are introduced in $\partial\phi/\partial t$ if neglecting the variation of the Coriolis parameter. Let us for example assume that u increases with 10 m/sec over an area of $5 \cdot 10^6$ km^2. Putting $f_y/f = 2 \cdot 10^{-7}$ m^{-1} we obtain an error in evaluating the circulation around this area which corresponds to a velocity of about 2 m/sec along the periphery. This velocity field then induces systematic errors in the vorticity advection.

The velocity field given by (5.20) is nondivergent, the geostrophic wind is not. It is of course questionable if the assumption of nondivergence is permissible, but if we want to remove this assumption it certainly must involve more than merely estimating the divergence geostrophically (cf. 5 d). It is therefore advantageous to apply this assumption strictly. The velocity field to be added to the geostrophic one to make it nondivergent is easily found by introducing the notations

$$\left.\begin{aligned} u &= -\frac{1}{f}\frac{\partial\phi}{\partial y} + \frac{1}{f}\frac{\partial\alpha}{\partial x} \\ v &= \frac{1}{f}\frac{\partial\phi}{\partial x} + \frac{1}{f}\frac{\partial\alpha}{\partial y} \end{aligned}\right\} \tag{5.33}$$

where $\qquad \nabla^2\alpha = f_y v$

The intensity of this additive velocity component is of the same order of magnitude as the one mentioned in the previous paragraph.

We shall not prolong this discussion, as the importance of the different terms in (5.21) is best illustrated by making a forecast using the correctly computed stream-function.

If the equation is elliptic in its type one can solve for ψ from a knowledge of ϕ by relaxation methods. The criterion for ellipticity depends upon the solution itself (5.24), but if neglecting the variation of the Coriolis parameter it is only a function of the given values of ϕ and f, (5.25). To avoid having to investigate if this criterion is satisfied at every step in the iterative process and to investigate the importance of the various terms separately the terms depending on f_x and f were neglected in the following considerations. Let us assume we have an approximate solution ψ'. Denoting the residual by ε we obtain

$$f\nabla^2\psi' + 2(\psi'_{xx}\psi'_{yy} - \psi'^2_{xy}) - \nabla^2\phi = \varepsilon \tag{5.34}$$

Now vary the function ψ' in order to decrease the residual ε successively. We get

$$\left.\begin{aligned} (f + 2\psi'_{yy})\,\delta\psi'_{xx} - 4\psi'_{xy}\delta\psi'_{xy} + \\ + (f + 2\psi'_{xx})\,\delta\psi'_{yy} = \delta\varepsilon \end{aligned}\right\} \tag{5.35}$$

This is a linear elliptic differential equation in $\delta\psi'$ if (5.25) is satisfied. By using the approximate solution ψ' we can estimate this correction $\delta\psi'$ by solving this linear equation by relaxation methods. Introducing finite differences for the evaluation of the derivatives of $\delta\psi'$ (5.35) becomes

$$\begin{aligned} &\left[f_{ij} + 2\left(\frac{\partial^2\psi'}{\partial y^2}\right)_{i,j}\right](\delta\psi'_{i+1,j} + \delta\psi'_{i-1,j} - 2\delta\psi'_{i,j}) + \\ &+ \left[f_{ij} + 2\left(\frac{\partial^2\psi'}{\partial x^2}\right)_{i,j}\right](\delta\psi'_{i,j+1} + \delta\psi'_{i,j-1} - 2\delta\psi'_{i,j}) - \\ &- \left(\frac{\partial^2\psi'}{\partial x\partial y}\right)_{i,j}(\delta\psi'_{i+1,j+1} + \delta\psi'_{i-1,j-1} - \\ &- \delta\psi'_{i+1,j-1} - \delta\psi'_{i-1,j+1}) = \frac{\delta\varepsilon_{i,j}}{m^2_{i,j}} \end{aligned} \tag{5.36}$$

where

$$m^2_{ij} = \frac{g(\varphi_{ij})}{(\Delta s)^2} \tag{5.37}$$

Δs being the gridsize at a particular latitude. $g(\varphi_{ij})$ is a function of the latitude and depends upon the map projection used. At the point (i,j) (5.36) may be satisfied by putting

$$\delta\psi'_{ij} = -\frac{\delta\varepsilon_{ij}}{4m^2_{ij}(f_{ij} + \nabla^2_{ij}\psi')} \tag{5.38}$$

and

$$\left.\begin{aligned} \delta\psi'_{i+1,j+1} = \delta\psi'_{i,j+1} = \delta\psi'_{i-1,j+1} = \delta\psi'_{i+1,j} = 0 \\ \delta\psi'_{i-1,j} = \delta\psi'_{i+1,j-1} = \delta\psi'_{i,j-1} = \delta\psi'_{i-1,j-1} = 0 \end{aligned}\right\} \tag{5.39}$$

If now assuming that $\delta\varepsilon_{i,j} = -\varepsilon_{i,j}$ and that $\varepsilon_{i,j}$ is given by the finite difference form of (5.34) we have improved the solution in the point (i,j) and the new value $\psi''_{i,j}$ is

$$\psi''_{i,j} = \psi'_{i,j} + \delta\psi'_{i,j} \tag{5.40}$$

The correction thus obtained does not give a complete agreement for the new values at the point (i,j) because of the non-linear character of (5.34) and the linearization that is involved in the derivation. It is, however, no point in trying to correct for this now before the other points have been relaxed in the same way, because it is still only an approximation. Thus we proceed to the next point and in correcting the ψ'-value we make now use of the new and better value ψ'' in the points already treated. The method then becomes similar to the so called Liebmann-method. The convergence of an iterative process of this kind has been discussed by NIRENBERG (1953).

It was mentioned before that the criterion for ellipticity usually is not fulfilled everywhere on an ordinary 500-mb map. For example on October 2, 1954, 0300 GMT the inequality (5.25) was satisfied over about 94 % of the total area. The procedure outlined above breaks down in the cases when the equation becomes hyperbolic at some point. To be able to get an idea of the importance of the term it was therefore decided to change the value of $\nabla^2\phi$ in those points so that

$$\frac{\nabla^2\phi}{f} = -\alpha\frac{f}{2} \qquad (5.41)$$

where $\alpha = 0.97 < 1$. This is of course not going to be the ultimate method to be used, but will still be of some interest here. The most serious consequence of this modification of the data is that the circulation around the total area is changed somewhat.

The results of some forecasts using the stream function thus obtained will be reported in a following issue of Tellus. Some questions regarding the transformation to a stereographic projection will also be discussed.

c. *The effect of a non-vanishing divergence*

In the derivation of (5.18) as well as the ordinary forecast equation (5.7) it was assumed that $\omega(p_0) = \omega(p_1) = 0$, i.e. the horizontal divergence integrated over the vertical is equal to zero. In other words, we have used the non-divergent barotropic model. Also the considerations in 5 b apply to this case. The fact that the pressure changes at the surface of the earth only amount to a few per cent of the total pressure indicates that this assumption is permissible as a first approximation as shown by CHARNEY (1949). Extending the integration to the top of the atmosphere ($p_1 = 0$) he derived the expression

$$\frac{\partial\bar{\zeta}}{\partial t} + \mathbf{v}\cdot\nabla(K\bar{\zeta} + f) = \frac{f}{p_0}\frac{\partial p_0}{\partial t} \qquad (5.42)$$

where p_0 is the surface pressure. (Here again it has been assumed $\zeta \ll f$).

Even if this term is small it is of principal importance as has been shown by ROSSBY (1945) and YEH (1949). For such a demonstration it is again feasible to study an idealized model of the atmosphere, i.e. a homogeneous and incompressible fluid, however, now with a free surface to permit the divergence to be different from zero. They found that the divergence term determines the character of dispersion in such fluid and Yeh also attempted to account for the existence and the permanence of blocking waves. The importance of this term is mainly noticeable for very long waves.

The question is, however, if this model takes the divergence into consideration in the way it appears in the real atmosphere. We are interested in studying the changes of the flow-pattern within the troposphere and the vertical integration should therefore be extended over the troposphere only. We now know that the vertical velocities at the tropopause are quite systematic in that cyclones have a low tropopause and anticyclones a high one. These variations easily amount to 200 mb indicating that *the horizontal divergence or convergence within the troposphere as a whole may change the depth of the troposphere with values up to about 20 %*. This effect is approximately depicted by modifying the barotropic model in the following way: On top of the homogeneous and incompressible fluid layer, which is supposed to represent the troposphere, and is in motion accordingly, we place another fluid layer at rest with a density chosen in such a way that the vertical motion of the interface is of the same relative magnitude as the motion of the tropopause in reality. The motion induced in the upper fluid can be assumed to be small by letting the depth of it approach infinity. The motion in the lower fluid then is governed by the equations

$$\left.\begin{aligned} \frac{du}{dt} - fv &= -\varkappa g\frac{\partial D}{\partial x} \\ \frac{dv}{dt} + fu &= -\varkappa g\frac{\partial D}{\partial y} \end{aligned}\right\} \qquad (5.43)$$

$\varkappa = (\varrho - \varrho')/\varrho$, where ϱ and ϱ' are the densities of the lower and upper fluid respectively. The vorticity equation (5.2) is not changed but we are not permitted to put the divergence equal to zero. We have instead to relate it to the changes of the depth of the fluid as given by the continuity equation in (5.1). To get a crude idea of the importance of the divergence term we shall first make the same approximations as used in deriving the forecast equation (5.7) except for putting div $\mathbf{v} = 0$. This is of course inconsistent in certain respects. We shall return to a more exact formulation of the problem in section (d). For the time being we thus introduce the geostrophic approximation

$$\zeta = \frac{\varkappa g}{f} \nabla^2 D \tag{5.44}$$

The variation of f with latitude is neglected. For a given distribution of ζ we see that the total range for D is inversely proportional to $\varkappa$. For a typical wind pattern at 500 mb the total range of the height of the 500 mb surface is 600—800 m. The variations in the height of the tropopause amount to five or ten times this value. Changes of this order of magnitude at the interface between the two fluids in our model are obtained by putting $\varkappa \approx 8$. From (5.43) we now can derive the forecast equation

$$(\nabla^2 - \lambda^2) \frac{\partial D}{\partial t} = J(\eta, D) \tag{5.45}$$

where

$$\lambda^2 = \frac{f^2}{\varkappa g D_0} \tag{5.46}$$

It has here been assumed that the convective terms in the continuity equation can be neglected, which is true in case of exact geostrophic flow. The two terms on the left side of the equation (5.45) become equally important when the wave length L is given by $L = 2\pi\sqrt{\varkappa g D/f^2}$. For the values $\varkappa = 1/8$, $g = 10$ m sec^{-2}, $D_0 = 10^4$ m and $f = 10^{-4}$ sec^{-1} we obtain $L = 7{,}000$ km.

Yeh (1949) has studied the properties of waves in a barotropic model with a *free* surface and his results are directly transferable to this model provided (5.46) is used for evaluating λ. This modification is, however, very important. For example in applying the results to a study of blocking Yeh is forced to assume the solitary wave to be of very large scale to obtain significant results. The considerations above reduce the scale by a factor of about $\sqrt{\varkappa} \approx 3$ which brings the systems in very good agreement with the size of actually observed blocking ridges. This supports the idea that already the simple equation (5.45) would mean an improvement compared with the present forecast formula (5.7). Only the largest components of the flow are influenced by this change but in view of the fact that the results of the computations presented in section 3 seem to indicate that (5.7) describes the development of the very largest systems less accurately than middle sized disturbances it would be of interest to use eq. (5.45) in some actual forecasts. This is being planned.

d. *The interplay between the wind and pressure fields in the divergent case*

The treatment in the previous section was very approximate. The purpose was only to point out one principle effect of a non-vanishing divergence. It is now of some interest to extend the reasoning and to investigate in which way the more general divergence equation also here might replace the geostrophic approximation. In this way we shall arrive at a more general set of forecast equations for the barotropic model. Some comments on this problem have also been given by Charney (1955).

We shall make use of the following very important fact concerning the average motion of the atmosphere: *The horizontal divergence is one order of magnitude smaller than the relative vorticity.* This is obviously true if considering the average through the whole atmosphere, since the percentage variation of the pressure at the surface is small (10^{-6}sec^{-1}). Even if we only consider the troposphere and assume the tropopause to be a material surface, we find that the height variations of the tropopause in connection with disturbances in the troposphere give rise to an average horizontal divergence in the troposphere which is less than 10^{-5}sec^{-1}. The relative vorticity, on the other hand, is abont one order of magnitude larger.

The discussion here will be limited to the barotropic case and we shall use the model of a homogeneous and incompressible atmosphere with an other infinitely deep upper layer on top. The motion of such a fluid is governed by the equations (5.1) if replacing g by $\varkappa g$ where as before $\varkappa = (\varrho-\varrho')/\varrho$. ϱ and ϱ' are the densities of the lower and upper fluids respectively.

From the equations (5.1) we derive the vorticity equation and the divergence equation and obtain the following set of equations.

$$\frac{\partial \zeta}{\partial t}+\mathbf{v}\cdot\nabla(f+\zeta)+(f+\zeta)\operatorname{div}\mathbf{v}=0 \quad (5.47)$$

$$\frac{\partial}{\partial t}(\operatorname{div}\mathbf{v})+\mathbf{v}\cdot\nabla(\operatorname{div}\mathbf{v})+(\operatorname{div}\mathbf{v})^2-\zeta f-$$
$$-2\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right)+f_y u-f_x v+g\nabla^2 D=0 \quad (5.48)$$

$$\frac{\partial D}{\partial t}+\mathbf{v}\cdot\nabla D+D\operatorname{div}\mathbf{v}=0 \quad (5.49)$$

Putting div $\mathbf{v}=0$ the vorticity equation shows that the wind field completely determines its own development. The divergence equation is reduced to a relation between the pressure and wind fields not containing any time derivatives and the continuity equation becomes an identity.

A non-vanishing divergence influences the vorticity equation in two ways. The wind can no longer be represented by a stream function only, but the divergence field must be associated with a velocity field, which can be described by a velocity potential. Let us therefore put

$$\left.\begin{aligned} u&=-\frac{\partial\psi}{\partial y}+\frac{\partial\chi}{\partial x}\\ v&=\frac{\partial\psi}{\partial x}+\frac{\partial\chi}{\partial y}\end{aligned}\right\} \quad (5.50)$$

where

$$\left.\begin{aligned}\nabla^2\psi&=\zeta\\ \nabla^2\chi&=\operatorname{div}\mathbf{v}\end{aligned}\right\} \quad (5.51)$$

Secondly, the absolute vorticity is not conserved because of the divergence. However, since the vorticity is one order of magnitude larger than the divergence and the time-scale of atmospheric disturbances is one day or more both these effects are comparatively small. The major changes of the flow are already obtained by the non-divergent model as indicated by the comparatively successful results with this model. *It is therefore not necessary to know the divergence field with the same percentage accuracy as the vorticity field.* We therefore are permitted to neglect some of the terms in the equations (5.48) and (5.49), since these equations are used for obtaining the proper relation between the divergence field and the vorticity field. We shall later return to the question of the conditions under which these simplifications are permissable, as well as their physical meaning. Neglecting small terms in (5.48) we obtain the same balance equation between the pressure field and the wind field as in the non-divergent case.

$$g\nabla^2 D=\zeta f+2\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right)-f_y u+f_x v \quad (5.52)$$

while (5.49) will be used without any simplifications. The following procedure for numerical integration then is suggested: At a certain time $\tau\cdot\Delta t$ we know $\psi=\psi^\tau$, $\chi=\chi^\tau$ and $D=D^\tau$ and similarly at previous times. (Δt is the time step used in the integration.) Thus $(\partial\zeta/\partial t)^\tau$ can be determined from (5.47) and $\zeta^{\tau+1}$ is obtained by linear extrapolation

$$\zeta^{\tau+1}=\zeta^{\tau-1}+2\left(\frac{\partial\zeta}{\partial t}\right)^\tau\Delta t \quad (5.53)$$

We then obtain $\psi^{\tau+1}$ by solving the Poisson equation (5.51) relating ψ and ζ. This is done by relaxation method and the linear extrapolation from $\psi^{\tau-1}$ and ψ^τ gives a first guess. $\psi^{\tau+1}$ determines the non-divergent component of the wind field. We next want to determine the corresponding pressure field $D^{\tau+1}$ with the aid of (5.52). The major part of the wind field $\mathbf{v}^{\tau+1}$ is given by $\psi^{\tau+1}$ and the small component given by $\chi^{\tau+1}$ can be neglected, since we only need approximate values of D and still are able to forecast ψ with good accuracy. Thus the right side of (5.52) can be evaluated and $D^{\tau+1}$ is obtained by solving a Poisson equation for example by relaxation. Finally we want to get a better estimate of $\chi^{\tau+1}$ to be used in the next time-step. The wind and pressure fields and their changes imply a certain divergence field according to (5.49). Putting

$$\left(\frac{\partial D}{\partial t}\right)^{\tau+1/2}=\frac{1}{\Delta t}(D^{\tau+1}-D^\tau) \quad (5.54)$$

and evaluating $\mathbf{v}^{\tau+1/2}\cdot\nabla D^{\tau+1/2}$ as the mean value over the time interval between $\tau\cdot\Delta t$ and $(\tau+1)\Delta t$ we can evaluate $(\operatorname{div}\mathbf{v})^{\tau+1/2}$. With the aid of (5.51) we can determine $\chi^{\tau+1/2}$ and obtain finally $\chi^{\tau+1}$ by extrapolation

$$\chi^{\tau+1}=\chi^{\tau}+2\,(\chi^{\tau+1/2}-\chi^{\tau}) \qquad (5.55)$$

Thus we are ready for the next time step. Since we only need an approximate value of χ it may even be permissable to neglect the convective term i (5,49) as it is small in quasi-geostrohic flow.

At the initial time $\tau=0$ only D° and ψ° are known, the latter from the solution of the simplified divergence equation as outlined in the previous section. We therefore first have to determine χ°, which can be done approximately by neglecting the influence of the divergence in (5,47). Thus we can get an approximate value of $(\partial\zeta/\partial t)^{\circ}$ and thus also of $(\partial\psi/\partial t)^{\circ}$. With the aid of (5.52) we can evaluate $(\partial D/\partial t)^{0}$ and then solve for χ^{0} from (5.49). The computations can then start as described previously except for the fact that uncentered differences have to be used.

The scheme outlined here is possible only if ψ represents the major part of the wind field and the wind field corresponding to χ merely is a comparatively small correction.

Forecasts with this method are being prepared and the results will be reported in a following issue of Tellus. The details in the computational procedure will then also be given.

We shall give a few additional comments on the approximations made. The most serious one is the neglection of the time-dependant term $d(\operatorname{div}\mathbf{v})/dt$ in (5.48). Hereby all gravity-inertia oscillations are eliminated and a balanced state between pressure and wind is assumed. We can get an approximate idea of the importance of these processes in the following way:

Since $(\operatorname{div}\mathbf{v})$ is small, $(\operatorname{div}\mathbf{v})^2$ certainly can be neglected in (5.48). Furthermore we replace the total derivative $d(\operatorname{div}\mathbf{v})/dt$ by the local one neglecting the convective term $\mathbf{v}\cdot\nabla(\operatorname{div}\mathbf{v})$. We shall see that maintaining the local time derivatives in the equation permits the existance of gravity-inertia waves the speed of which (c) is about 100 m/s. As long as c is large compared with $\mathbf{v}$ these waves are fairly well described even if neglecting the convective term. Under all circumstances it means less simplifications than merely using the balance equation (5.52). We shall furthermore disregard the convective term in the continuity equation. This term is zero if the wind is parallell with the isobars and therefore as long as the departures from geostrophic or gradient flow are small this simplification is permissable. Finally we assume that the percentage variations of D are small and replace D by D_0. The continuity equation becomes

$$\frac{1}{D_0}\cdot\frac{\partial D}{\partial t}+\operatorname{div}\mathbf{v}=0 \qquad (5.56)$$

With these simplifications combination of (5.48) and (5.56) for elimination of $\partial D/\partial t$ gives

$$\frac{\partial^2}{\partial t^2}(\operatorname{div}\mathbf{v})-\varkappa g D_0\nabla^2(\operatorname{div}\mathbf{v})+f\,(f+\zeta)\operatorname{div}\mathbf{v}=$$
$$=\mathbf{v}\cdot\nabla(\zeta+f)+\frac{\partial}{\partial t}\left[2\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right)-f_y u+f_x v\right] \qquad (5.57)$$

Let us assume that the changes of the wind field are known over a certain time interval. We can then consider (5.57) as a wave-equation with a forcing term which is a function of these variations of the wind field and div $\mathbf{v}$ is determined if proper initial conditions are prescribed. It is obvious that this viewpoint is possible only if the divergence is small so that the right side of equation (5.57) is practically independant of it. In other words the "feed back" on the forcing term from the divergence can be disregarded. The homogeneous part of this equation is the same equation (generalized to two dimensions in space) as CAHN (1945) obtained for the adjustment of the pressure and wind fields to each other. The solution of (5.57) is composed of two parts, one being the solution of the homogeneous equation with certain initial values of div $\mathbf{v}$ and $\partial(\operatorname{div}\mathbf{v})/\partial t$ and another being the solution of the inhomogeneous equation with the initial conditions $\operatorname{div}\mathbf{v}=\partial(\operatorname{div}\mathbf{v})/\partial t=0$. The solution of the homogeneous equation is in principle the same as given by Cahn and takes care of the adjustment of an initial out of balance between the pressure and wind fields. The other part of the solution gives the steady adjustment of the pressure field because of the changing wind field.

CHARNEY (1955) points out that the use of the geostrophic approximation as initial conditions in the primitive equations of motion gives rise to large oscillations. It is clear from (5.57) that this must be the case. The geostrophic wind has a small divergence div $\mathbf{v}_g = v \cdot a^{-1} \cot \varphi$, where v is the wind component south—north and a is the radius of the earth. Neglecting $(\text{div}\,\mathbf{v})^2$ and $\mathbf{v} \cdot \nabla$ div $\mathbf{v}$ in (5.48) as before we furthermore obtain at the initial instant

$$\frac{\partial}{\partial t}(\text{div}\,\mathbf{v}) = 2\left(\frac{\partial u_g}{\partial x}\cdot\frac{\partial v_g}{\partial y} - \frac{\partial u_g}{\partial y}\cdot\frac{\partial v_g}{\partial x}\right) = $$
$$= \nabla \cdot (\mathbf{v}_g \cdot \nabla\, \mathbf{v}_g) \qquad (5.58)$$

This is also different from zero and has large values in areas of strong curvature of the flow, or if the deformation field is strong. The solution of the homogeneous equation then is a large amplitude oscillation and is a reflection of the adjustment of the wind and pressure fields towards an equilibrium characterized by (5.52). Relating the wind and pressure fields to each other by (5.52) initially means that we put div $\mathbf{v} = \partial$ (div $\mathbf{v})/\partial t = 0$ and no oscillations of this kind appear.

The equation (5. 52) has a maximum speed of progagation of influences which is equal to $c_0 = \sqrt{\varkappa \cdot g D_0}$. With the values previously assigned to $\varkappa$ and D_0 we get $c_0 = 400$ km h^{-1}. A change of the wind field at one point does not influence the pressure field beyond a distance of about 600 km in $1^1/_2$ hour which is the time step used in most forecasts presented here. The use of the balance equation (5.52) on the other hand means that the pressure immediately is influenced over the whole area considered.

In the derivation of (5.57) we made some approximations in the continuity equation which were not necessary if using the balance equation (5.52). It is therefore questionable if any improvements would be obtained by using this equation. However, it would be desirable to investigate this in some more detail.

The discussion has here been restricted to a barotropic model of the atmosphere. Since c_0 is so large and the divergence small in this model, the refinements in this last section will probably not improve the forecasts essentially. It is, however, of some interest to have these processes clarified in this comparatively simple case, as the importance of gravity-inertia waves probably is greater in the baroclinic case. Here we the encounter other difficulties. The speed of the internal waves, that may exist, is smaller than the influence speed given above (cf. BOLIN 1953 b). Furthermore ROSSBY (1938) already pointed out that in the case of a one layer fluid the wind field is very little changed during the adjustment, but in a two layer fluid a considerable loss of kinetic energy takes place. This is an indication of the fact that in a stratified fluid the adjustment of a velocity field varying along the vertical is associated by a considerable redistribution of the mass field. The horizontal divergence is then not so small compared with the vorticity any longer and the procedure outlined above becomes less accurate. It is then questionable if any balance equation between the wind and the pressure fields will describe the actual processes sufficiently well.

REFERENCES

BOLIN, B., 1953 a: Multiple-parameter models of the atmosphere for numerical forecasting purposes. *Tellus* **5**, pp. 207—218.

— 1953 b: The adjustment of a non-balanced velocity field towards geostrophic equilibrium in a stratified fluid. *Tellus* **5**, pp. 373—385.

BOLIN, B., CHARNEY, J., 1951: Numerical tendency computations from the barotropic vorticity equation. *Tellus*, **3**, pp. 248—257.

CAHN, A., 1945: An investigation of the free oscillations of a simple current system. *J. of Meteorology*, **2**, pp. 113—119.

CHARNEY, J., 1949: On the physical basis for numerical prediction of large-scale motions in the atmosphere. *J. of Meteorology*, **6**, pp. 371—385.

CHARNEY, J., 1955: The use of the primitive equations of motion in numerical forecasting. *Tellus* **7**, pp. 22—26.

CHARNEY, J., PHILLIPS, N., 1953: Numerical integration of the quasi-geostrophic equations of motion for barotropic and simple baroclinic flows. *J. of Meteorology*, **10**, pp. 71—99.

CLAPP, PH. F., 1953: Application of barotropic tendency equation to medium range forecasting. *Tellus* **5**, pp. 80—94.

COURANT, R., und HILBERT, D., 1931: *Methoden der mathematischen Physik*, Berlin.

EADY, E., 1952: Note on weather computing and the so-called $2^1/_2$ dimensional model. *Tellus* **4**, pp. 157—167.

ELIASSEN, A., 1952: Simplified models of the atmosphere, designed for the purpose of numerical weather prediction, *Tellus* **4**, pp. 145—156.

FJØRTOFT, R., 1952: On a numerical method of integrating the barotropic vorticity equation. *Tellus* **4**, pp. 179—194.

NIRENBERG, L., 1953: On non-linear elliptic partial differential equations and hölder continuity. *Communications on pure and applied mathematics*, **6**, pp. 103—156.

PETTERSSEN, S., 1953: On the relation between vorticity deformation and divergence and the configuration of the pressure field. *Tellus* **5**, pp. 231—238.

ROSSBY, C.-G., 1938: On the mutual adjustment of pressure and velocity distribution in aertain simple current systems. II. *J. of Marin. Res.*, **1**, 239—263.

— 1945: On the propagation of frequencies and energy in certain types of oceanic and atmospheric waves. *J. of Meteorology*, **2**, pp. 187—204.

— 1949: On a mechanism for the release of potential energy in the atmosphere. *J. of Meteorology*, **6**, pp. 163—180.

ROSSBY, C.-G., and collaborators, 1939: Relation between variations in the intensity of the zonal circulation of the atmosphere and the displacements of the semi-permanent centers of action, *J. of Marine Research*, **2**, pp. 38—55.

SCHERHAG, R., 1948: *Wetteranalyse und Wetterprognose*, Berlin, Springer-Verlag. 424 pp.

SHERMAN, L., 1952: On the scalar-vorticity and horizontal divergence equations. *J. of Meteorology*, **9**, pp. 359—366.

STAFF MEMBERS, INSTITUTE OF METEOROLOGY UNIVERSITY OF STOCKHOLM, 1954: Results of forecasting with the barotropic model on an electronic computor (BESK), *Tellus* **6**, pp. 139—149.

YEH, T. C., 1949: On energy dispersion in the atmosphere. *J. of Meteorology*, **6**, pp. 1—16.