Formally accounting for the existence of unresolved motion

A naive example of model "dynamics": heat advection

$$\frac{\partial \theta}{\partial t} = -\vec{U} \cdot \nabla \theta$$

- Grid spacing for today's NWP down to order 10 km
- This leaves many scales of motion unresolved
- Implication? – effects of unresolved motion must be parameterized
- Starting point to understand those effects is the Reynolds decomposition
- Let $f = f(x, y, z, t)$ and $g = g(x, y, z, t)$ be arbitrary properties
- Divide every dependent variable into the sum of a resolved part (which we can regard as a volume average over a grid cell) and a residual (the "fluctuation"):

$$f = \overline{f}(x, y, z, t) + f\,'(x, y, z, t)$$

- Many notations used for the "mean" (resolved part), e.g. $\overline{f}\,,\ F\,,\ \langle f \rangle$
- Here I'll use the overbar
- At best, only *statistics* of the unresolved fields *f* ' , *g* ' will be available in NWP – and those estimated by a semi-empirical "*parameterization*"

- Formally, a volume average looks like

$$\bar{f}(x,y,z,t)=\frac{1}{L_x L_y L_z}\int_{x-(L_x)/2}^{x+(L_x)/2}\int_{y-(L_y)/2}^{y+(L_y)/2}\int_{z-(L_z)/2}^{z+(L_z)/2} f(x_*,y_*,z_*,t)\ dx_*\, dy_*\, dz_*$$

- Here x_* (etc.) are dummy variables; L_x would be the gridlength along x. Implicitly, each NWP gridpoint value is an average over a gridcell of volume $L_x L_y L_z$
- Still a function of (x,y,z,t), but presumably a much smoother function than f
- Averaging a product leads to a surprising result: the first step is easy

$$f=\bar{f}+f'$$

$$g=\bar{g}+g'$$

$$f\ g=[\bar{f}+f'][\bar{g}+g']=\bar{f}\ \bar{g}+\bar{f}\ g'+f'\ \bar{g}+f'\ g'$$

- We simplify this using Reynolds' averaging rules

$$\overline{f+g} = \overline{f} + \overline{g}$$

$$\overline{\alpha f} = \alpha \overline{f}$$ (α any constant)

$$\overline{\frac{\partial f}{\partial s}} = \frac{\partial \overline{f}}{\partial s}$$ ("s" being x or y or z or t)

$$\overline{\overline{f}\, g} = \overline{f}\ \overline{g}$$

<u>STATISTICS</u> of f

Mean square value : $\overline{f^2}$

Variance : $\sigma_f^2 \equiv \overline{(f\,')^2}$

Standard deviation: $\sigma_f \equiv \sqrt{\overline{(f\,')^2}}$

imply

$$\overline{\overline{f}} = \overline{f} \qquad \overline{g\,'} = 0 \qquad \overline{\overline{f}\, g\,'} = \overline{f}\ \overline{g\,'} = 0 \qquad \overline{\overline{f}\ \overline{g}} = \overline{f}\ \overline{g}$$

$$\overline{f\ g} = \overline{[\overline{f} + f\,'][\overline{g} + g\,']} = \overline{\overline{f}\ \overline{g} + \overline{f}\ g\,' + f\,'\ \overline{g} + f\,'\ g\,'}$$

$$= \overline{f}\ \overline{g} + \overline{\overline{f}\ g\,'} + \overline{f\,'\ \overline{g}} + \overline{f\,'\ g\,'} = \overline{f}\ \overline{g} + \overline{f\,'\ g\,'}$$

"covariance"

- Recall the instantaneous, local vertical convective flux density of humidity is

$$E = w\,\rho_v \quad \left[\mathrm{kg\ m^{-2}\ s^{-1}}\right]$$

- The mean vertical vapour flux density is therefore

$$\overline{E} = \overline{w}\,\overline{\rho}_v + \overline{w'\,\rho_v'}$$

- First term is vapour transport by the resolved flow; second, by the unresolved flow. Multiplying by the latent heat of vapourization we get the latent heat flux density Q_E, i.e. $Q_E = L_v\,\overline{E}$

- Similarly, $\rho\,c_p T \;\left[\mathrm{J\ m^{-3}}\right]$ being the volumetric content of sensible heat, the mean convective vertical flux density of sensible heat is

$$Q_H = \rho\,c_p\,\left[\overline{w}\,\overline{T} + \overline{w'\,T'}\right]$$

where the $\rho\,c_p$ product has been treated as (locally) constant

- It remains to show the manner in which the covariances alter evolution of $\overline{\rho_v}$, $\overline{T}$

- The streamwise momentum per unit volume is: $\rho\, u \quad [\mathrm{kg\,m\,s^{-1}\,m^{-3}}]$

- Multiply by *w* to get the vertical flux density of streamwise momentum: $\rho\, u\, w$

- Mean (convective) vertical momentum flux density is therefore

$$\tau_{xz} = \rho \bar{u}\, \bar{w} + \rho \overline{u'w'}$$

- Adding a new adjective and dropping others, the “kinematic” momentum flux is

$$\tau_{xz}/\rho = \bar{u}\, \bar{w} + \overline{u'w'}$$

  where the second term is called a “Reynolds stress”

- The Reynolds stress tensor R_{ij} is the matrix

$$\begin{bmatrix} \overline{u'^2} & \overline{u'v'} & \overline{u'w'} \\ \overline{v'u'} & \overline{v'^2} & \overline{v'w'} \\ \overline{w'u'} & \overline{w'v'} & \overline{w'^2} \end{bmatrix}$$

- This can also be called the velocity covariance tensor. Diagonal elements are the velocity variances, $\sigma_u^2 \equiv \overline{u'^2}$ (etc.), where σ_u is the standard deviation of *u*

- Let's start with an equation for conservation of energy. To keep life simple, let's assume a dry system and that the motion is adiabatic. Then

$$\frac{D\theta}{Dt} \equiv \frac{\partial\theta}{\partial t} + \vec{u}\cdot\nabla\theta = 0$$

- Let's further assume the velocity is non-divergent, $\nabla\cdot\vec{u}=0$. Then since $\nabla\cdot(\vec{u}\,\theta)=\vec{u}\cdot\nabla\theta+\theta\,\nabla\cdot\vec{u}=\vec{u}\cdot\nabla\theta$ we can put the heat equation in “flux form” as

$$\frac{D\theta}{Dt} \equiv \frac{\partial\theta}{\partial t} + \nabla\cdot(\vec{u}\,\theta) = \frac{\partial\theta}{\partial t} + \frac{\partial u\theta}{\partial x} + \frac{\partial v\theta}{\partial y} + \frac{\partial w\theta}{\partial z} = 0$$

- Now applying the Reynolds averaging rules,

$$\frac{\partial\overline{\theta}}{\partial t} + \frac{\partial\overline{u\theta}}{\partial x} + \frac{\partial\overline{v\theta}}{\partial y} + \frac{\partial\overline{w\theta}}{\partial z} = 0$$

- Expand each product,

$$\frac{\partial \overline{\theta}}{\partial t} + \frac{\partial}{\partial x}\left[\overline{u}\,\overline{\theta} + \overline{u'\theta'}\right] + \frac{\partial}{\partial y}\left[\overline{v}\,\overline{\theta} + \overline{v'\theta'}\right] + \frac{\partial}{\partial z}\left[\overline{w}\,\overline{\theta} + \overline{w'\theta'}\right] = 0$$

- To make this look more familiar, convert back to advection form

$$\frac{D\overline{\theta}}{Dt} \equiv \frac{\partial \overline{\theta}}{\partial t} + \overline{u}\,\frac{\partial \overline{\theta}}{\partial x} + \overline{v}\,\frac{\partial \overline{\theta}}{\partial y} + \overline{w}\,\frac{\partial \overline{\theta}}{\partial z} = -\frac{\partial \overline{u'\theta'}}{\partial x} - \frac{\partial \overline{v'\theta'}}{\partial y} - \frac{\partial \overline{w'\theta'}}{\partial z}$$

- Finally, we see the influence of the unresolved processes on resolved (mean) θ. Gradients in the unresolved heat fluxes (“eddy heat flux divergence-convergence”) “drive” an evolution of the volume-average temperature. The vertical eddy heat flux is particularly important, and especially within the turbulent ABL
- compare this with the naive "pure dynamics" treatment (eqn on page 1)

- Recall the instantaneous *u*-mtm eqn is

$$\frac{Du}{Dt} \equiv \frac{\partial u}{\partial t} + \vec{u} \cdot \nabla u = \frac{-1}{\rho} \frac{\partial p}{\partial x} + f\, v + \nu \nabla^2 u$$

"molecular friction"

- Neglecting molecular friction and adopting flux form (under the Boussinesq approximation, density treated as constant and velocity assumed non-divergent),

$$\frac{\partial u}{\partial t} = \frac{-1}{\rho_0} \frac{\partial p}{\partial x} + f\, v - \nabla \cdot (\vec{u}\, u) = \frac{-1}{\rho_0} \frac{\partial p}{\partial x} + f\, v - \frac{\partial uu}{\partial x} - \frac{\partial vu}{\partial y} - \frac{\partial wu}{\partial z}$$

- After applying the Reynolds average, and bringing advection terms to the lhs,

$$\frac{\partial \overline{u}}{\partial t} + \overline{u} \frac{\partial \overline{u}}{\partial x} + \overline{v} \frac{\partial \overline{u}}{\partial y} + \overline{w} \frac{\partial \overline{u}}{\partial z} = \frac{-1}{\rho_0} \frac{\partial \overline{p}}{\partial x} + f\, \overline{v} - \frac{\partial \overline{u'u'}}{\partial x} - \frac{\partial \overline{v'u'}}{\partial y} - \frac{\partial \overline{w'u'}}{\partial z}$$

"turbulent friction"

- Gradients in the unresolved (or "eddy") momentum fluxes, i.e. Reynolds stress gradients, result in acceleration (or deceleration) of the resolved velocity

- Will illustrate only the so called "first-order" closure. Taking the vertical eddy flux of heat as an example, by analogy with Fourier's law of conduction we write

$$\overline{w'\theta'} = -K_h \frac{\partial \overline{\theta}}{\partial z}$$

where K_h [$m^2\ s^{-1}$] is the "eddy diffusivity for heat" (very much larger than the molecular thermal diffusivity). The eddy flux is "driven" by the vertical gradient in the resolved (or "mean") potential temperature

- So what is the eddy heat flux in a neutral layer?
- This "parameterization" leaves us with the problem of how to prescribe K_h
- Some empiricism is inevitable
- Analogous closure for Reynolds stresses of form $\overline{u'w'} = -K_m \left[\frac{\partial \overline{u}}{\partial z} + \frac{\partial \overline{w}}{\partial x} \right]$

where K_m [$m^2\ s^{-1}$] is the "eddy viscosity" (often set equal to K_h)

- Varies in proportion to *z* near ground
- Increases with increasing mean wind shear
- Increases with increasing thermal instability
- Becomes small at the top (δ) of the ABL