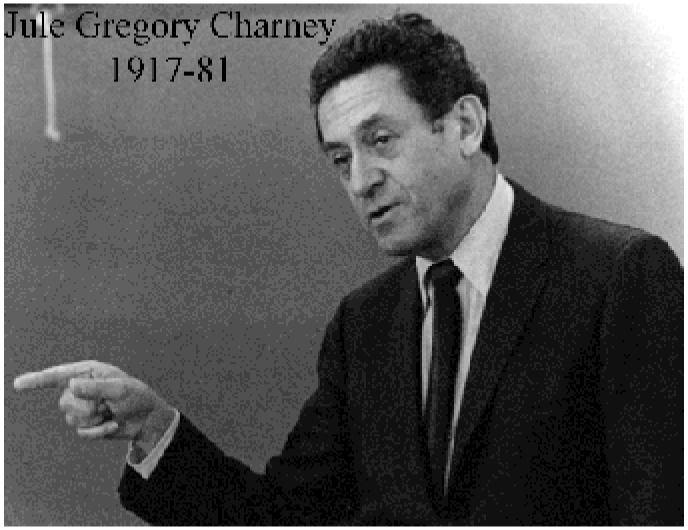
Motivation:

- "forecasters do not rely exclusively on computer-generated guidance... they attempt to understand the factors likely to be responsible for the current and the next day's weather." (Durran & Snellman, 1987)
- weather forecasting is largely synonymous with the ability to anticipate how the isobaric height field will *develop*, but this is not addressed by geostrophic theory
- the QG eqns predict the rates of change of the height, vertical velocity and vertical vorticity; they are inexact, but provide a useful interpretive theory
- the QG eqns are useful because in mid-latitude synoptic scale systems "the fields of vertical motion (ω) and geopotential tendency ($\chi = g\, \partial Z/\partial t$) are primarily determined by the distribution of vorticity advection and thermal advection."
- QG theory reduces the 5-variable primitive equations to a 1-equation system: all variables (u_g, v_g, ω, p, T) can be obtained from the height field Z. Early NWP models exploited this.

Derivation:

- Rossby, Eady, Charney, Phillips and others: e.g. Charney 1949, On a Physical Basis for Numerical Prediction of Large-Scale Motions in the Atmosphere, J. Meteorol. 6, 371-385.
- emerges from scale analysis; or by an asymptotic expansion of the governing eqns in the Rossby number $R_o = U/(f\,L)$, *U* and *L* being velocity and length scales characterizing synoptic scale motion



Assumptions/Restrictions/Idealizations:

- adiabatic, frictionless, extra-tropical flow
- hydrostatic approx.
- *β*-plane* approx., $f = f(y) = f_0 + \beta\, y$

$$f_0 = 2\,\Omega \sin\varphi_0$$

$$\beta = (\partial f / \partial y)_{\varphi_0} = 2\,\Omega \cos\varphi_0 / R$$

**as distinct from the f-plane approximation, wherein f is assumed to be constant (AMS Glossary)*

- decompose horiz. velocity field: $\vec{V} = \vec{V}_g + \vec{V}_{ag}$ where $\vec{V}_g = (u_g, v_g) = \dfrac{g}{f_0}\,\hat{k} \times \nabla Z$
- neglect vertical advection
- evaluate horizontal advection using the *geostrophic* component (only)

Some key variables/features/definitions:

$$\vec{V}_g = \frac{g}{f_0}\hat{k}\times\nabla Z = \frac{g}{f_0}\left(-\frac{\partial Z}{\partial y}, \frac{\partial Z}{\partial x}\right)$$

(Z , or gZ/f_0, constitutes a "streamfunction")

$$\nabla_H\cdot\vec{V}_g = \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = \frac{g}{f_0}\left(-\frac{\partial^2 Z}{\partial x\partial y} + \frac{\partial^2 Z}{\partial y\partial x}\right) = 0$$

(only the ageostrophic component contributes to horiz. divergence)

$$\eta = \zeta_g + f_0 + \beta y$$

vertical component of the absolute vorticity

$$\zeta_g = \hat{k}\cdot(\nabla_H\times\vec{V}_g) = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{g}{f_0}\left(\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2}\right) = \frac{g}{f_0}\nabla_H^2 Z$$

(Poisson's eqn., easily solved numerically. Vertical vorticity equals Laplacian of the streamfunction. Z is "invertible" to give the vorticity and the wind)

$$\frac{D_g}{Dt} \equiv \frac{\partial}{\partial t} + \vec{V}_g\cdot\nabla_H$$

material derivative following the geostrophic wind

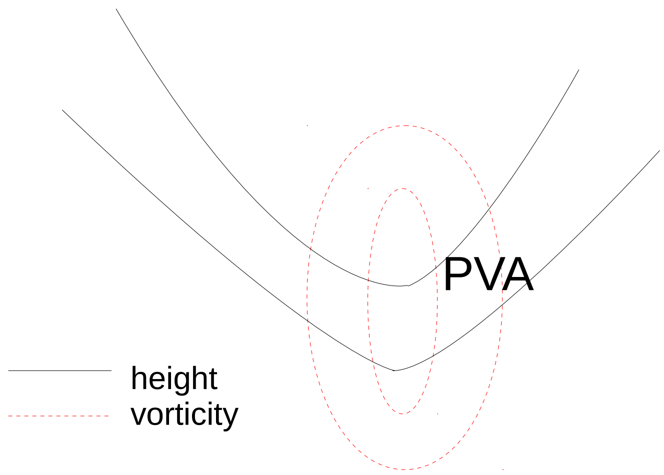
$$\frac{\partial \zeta_g}{\partial t} + \vec{V}_g \cdot \nabla_H (\zeta_g + f) = f_0 \frac{\partial \omega}{\partial p}$$

where $$\nabla_H \cdot \vec{V}_{ag} + \frac{\partial \omega}{\partial p} = 0$$

Alternative statement:

$$\frac{D_g \eta}{Dt} = -f_0 \nabla \cdot \vec{V}_{ag}$$

absolute vorticity of a parcel varies along its path, as driven by the forcing term on the r.h.s., which represents horizontal convergence/divergence (or equivalently, vertical stretching of the column)



- At the trough axis, local max in cyclonic relative vorticity and thus in abs. vorticity
- As a parcel moves downwind from the trough axis, its vorticity decreases, i.e. $D_g\eta/Dt < 0$, implying $\nabla \cdot \vec{V}_{ag} > 0$
- Using the natural coords, value of $-v \frac{\partial \eta}{\partial s}$ is positive ("PVA")

Vorticity advection signifies horizontal divergence, which is an indicator for lift

$$\left[\nabla_H^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\omega = \frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\vec{V}_g \cdot \nabla_H \eta\right] + \frac{R_d}{\sigma\, p}\nabla_H^2\left[\vec{V}_g \cdot \nabla_H T\right]$$

$\propto$ height gradient of vorticity advection

$\propto$ Laplacian of temperature advection

where $\sigma = -\frac{R_d T}{p}\frac{\partial \ln\theta}{\partial p}$ $[\mathrm{Pa}^{-2}\,\mathrm{m}^2\,\mathrm{s}^{-2}]$ is the "static stability" (normally positive and of order 10^{-6} in free atmos.)

If $\sigma \rightarrow \infty$, $\nabla_H^2\,\omega = 0$ (Laplace's eqn) and if ω=0 on boundaries, then ω=0 everywhere

Recall, we define the advective "temperature advection rate" as $A_T = -\vec{V}_g \cdot \nabla_H T$

and, $\omega = -\rho\, g\, w$. We can multiply through by -1 to get

$$\rho\, g\left[\nabla_H^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right] w = \frac{f_0}{\sigma}\frac{\partial A_\eta}{\partial p} + \frac{R_d}{\sigma\, p}\nabla_H^2 A_T$$

We'd have to call this the quasi-geostrophic "w" equation (double-u), rather than "omega" eqn

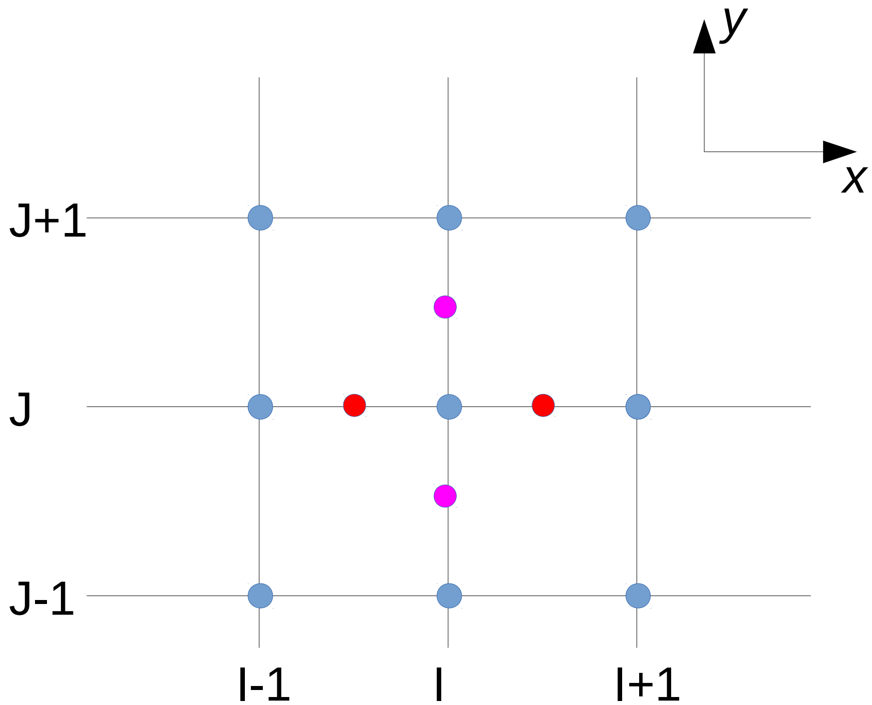
height gradient of vorticity advection

Laplacian of temperature advection

In Cartesians, $\nabla_H^2 \omega = \dfrac{\partial^2 \omega}{\partial x^2} + \dfrac{\partial^2 \omega}{\partial y^2}$

And recall, curvature is slope of slope

$$\frac{\partial^2 \omega}{\partial x^2} \equiv \frac{\partial}{\partial x}\frac{\partial \omega}{\partial x}$$



Evaluate *∂ω/∂x* at red points by finite difference, using known values on the grid:

$$\left(\frac{\partial \omega}{\partial x}\right)_{I-1/2,J} = \frac{\omega_{I,J} - \omega_{I-1,J}}{\Delta x} \qquad \left(\frac{\partial \omega}{\partial x}\right)_{I+1/2,J} = \frac{\omega_{I+1,J} - \omega_{I,J}}{\Delta x}$$

$$\left(\frac{\partial^2 \omega}{\partial x^2}\right)_{I,J} = \frac{\dfrac{\omega_{I+1,J} - \omega_{I,J}}{\Delta x} - \dfrac{\omega_{I,J} - \omega_{I-1,J}}{\Delta x}}{\Delta x} = \frac{\omega_{I+1,J} + \omega_{I-1,J} - 2\omega_{I,J}}{\Delta x^2}$$

Similarly, evaluate *∂ω/∂y* at magenta points by finite difference – and if $\Delta x = \Delta y = \Delta$, then

$$\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)_{I,J} = \frac{\omega_{I+1,J} + \omega_{I-1,J} + \omega_{I,J+1} + \omega_{I,J-1} - 4\omega_{I,J}}{\Delta^2}$$

Laplacian is positive if central value smaller than average of its neighbours

$$\rho g \left[\nabla_H^2 + \frac{f_0^2}{\sigma} \frac{\partial^2}{\partial p^2} \right] w = \frac{f_0}{\sigma} \frac{\partial A_\eta}{\partial p} + \frac{R_d}{\sigma \, p} \nabla_H^2 A_T$$

height gradient of vorticity advection

Laplacian of temperature advection

- interpret the LHS as (3D) Laplacian of vertical velocity, visualized with gridlength Δ

$$\rho g \left[\frac{\overline{w}_{\mathrm{nbrs}} - w}{\Delta^2} \right] = \frac{f_0}{\sigma} \frac{\partial A_\eta}{\partial p} + \frac{R_d}{\sigma \, p} \left[\frac{\overline{A}_{T\,\mathrm{nbrs}} - A_T}{\Delta^2} \right]$$

units are consistent

- if RHS is negative, central value of *w* is bigger than its neighbours (local updraft)**

- to make the temperature advection term negative, need A_T exceeding its neighbours –

  local "hot spot" for thermal advection implies updraft

- to make the vorticity advection term negative, we need A_η to decrease with increasing *p*,

  i.e. A_η to increase with increasing height – *strong PVA aloft implies updraft*

**opposite is true for the omega eqn

$$\left[\nabla_H^2 + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi = -f_0\,\vec{V}_g\cdot\nabla_H\eta - \frac{f_0^2 R_d}{\sigma\;p}\frac{\partial}{\partial p}\left[-\vec{V}_g\cdot\nabla_H T\right]$$

(f_0 times) vorticity advection

proportional to height derivative of temperature advection

where $\chi = g\dfrac{\partial Z}{\partial t}$. Again, LHS can be (loosely) interpreted as a 3D Laplacian of χ , such that

$$g\frac{\partial}{\partial t}\left[\frac{\overline{Z}_{\mathrm{nbrs}} - Z}{\Delta^2}\right] = f_0\,A_\eta + \frac{f_0^2 R_d^2 T}{\sigma\;p^2\;g}\;\frac{\partial A_T}{\partial z}$$

- *Z* falls relative to its neighbours if PVA is occurring, i.e. $A_\eta \equiv -\vec{V}_g\cdot\nabla_H\eta > 0$
- *Z* falls relative to its neighbours if warm advection increases with increasing with height

See qgforcing_maps.pdf for some examples. In that file, maps are given showing the RHS of the height tendency equation and of the *omega* equation. In interpreting those charts, recall that wherever the RHS of the omega eqn is positive, there is forcing for ascent – whereas for the *w* equation a negative RHS means ascent