The horizontally-homogeneous atmospheric surface layer - an intro. for EAS 471

J.D. Wilson, EAS, U. Alberta, Feb., 2007

For now we will define the surface layer ("ASL") to be a shallow layer adjacent to ground, with depth z_s of order $\delta/10$ where δ is the depth[1] of the entire Atmospheric Boundary Layer (ABL, also known as the 'friction layer'); in this ASL the vertical gradients in mean windspeed, temperature, and humidity are very strong, and substantial (turbulent, convective) fluxes of heat, momentum and moisture pass upward or downward, linking the surface to the atmosphere. Since the ASL is shallow, these vertical fluxes of heat, mass and momentum change only by a small fraction between ground and z_s, so sometimes the surface layer is called the 'constant flux layer'.

In the ABL the velocity field is (to a satisfactory level of approximation) non-divergent, and this is true of the instantaneous field, fluctuation field, and mean field. Now the mean velocity field being non-divergent, it follows at once from

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0 \tag{1}$$

$$\overline{w}(0) = 0 \tag{2}$$

that *if* flow statistics are independent of x, y then (for all z) $\overline{w}(z) = 0$. We can make $\overline{v} = 0$ at ground too, by choice of the coordinate orientation, but due to turning of the mean wind with height (due to the Coriolis force), $\overline{v}$ will not vanish aloft (though this turning is often neglected in the shallow "surface-layer").

'**Horizontal-homogeneity**' ('horizontally-uniformity') - *statistics* do not vary in the horizontal

'**Stationarity**' ('steady state') - *statistics* do not vary in time

The above simplifications are perhaps never 100% legitimate, but in adopting them we can at least conceptualize an ideal surface layer, and doing so proves very useful.

[1] Another common symbol for the depth of the ABL is 'z_i', referring to the height above ground of the base of a capping inversion (hence the 'i') that is a common (but not universal) feature of the daytime ABL.

Mean momentum equations

The mean momentum equations, obtained by Reynolds-averaging the Navier-Stokes equations[2], are:

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{u}_i\,\overline{u}_j + \overline{u_i'\,u_j'}\right) = -\frac{1}{\rho_0}\,\frac{\partial \overline{p}}{\partial x_i} + g\frac{\overline{T}}{T_0}\,\delta_{3i} - F_{ci} \tag{3}$$

(F_{ci} denotes the Coriolis force; I have neglected viscous momentum transport). Now if we simplify for the stationary, horizontally-homogeneous surface layer, the streamwise component is

$$\frac{\partial \overline{u'w'}}{\partial z} = -\frac{1}{\rho_0}\,\frac{\partial \overline{p}}{\partial x} + f\overline{v} \tag{4}$$

(e.g. Garratt, 1992, eqn 2.48) where f is the Coriolis parameter ($f = 2\Omega\sin\phi$, where ϕ is latitude). We have a balance of the pressure gradient, Coriolis and 'friction' forces, where 'friction' is expressed as the divergence of the turbulent momentum flux. We can orient our surface layer coordinate system so that $\overline{v} = 0$. In addition, we already know from continuity equation that $\overline{w} = 0$. So we have a single non-zero component of the mean velocity, $\overline{u}(z)$. Unfortunately this reprehensible actor refuses to appear in its own governing equation. Also, our equation contradicts the notion of the momentum flux being height independent - it can't be. Our 'constant stress layer' is a convenient fiction. Recall, though, that we are unable to satisfactorily measure the existent small height changes in these turbulent convective fluxes, and that neglecting the height changes is for many purposes acceptable.

Later we'll see how a theory for $\overline{u}(z)$ is developed, either by an intuitive scaling approach (MOST), or by heuristic introduction of a flux-gradient closure.

TKE budget assuming stationarity and horizontal uniformity

The existence of a deep friction layer is a consequence of vertical momentum transport[3] by turbulent vertical velocity fluctuations. In a turbulent region of the atmosphere, by definition,

[2]Under the Boussinesq approximation, which permits treating the velocity field as incompressible and the density as constant except where multiplied by gravity.

[3]If you like, of momentum *deficit*. The no-slip condition at ground means there is a streamwise velocity deficit at that level exactly equal to the Geostrophic (or free stream) windspeed; that velocity deficit is mixed aloft by the fluctuating vertical motion.

the turbulent kinetic energy (strictly, per unit mass)

$$k = \frac{1}{2}\left({\sigma_u}^2 + {\sigma_v}^2 + {\sigma_w}^2\right) \tag{5}$$

is non-zero. Turbulence dissipates kinetic energy, so unless supplied with energy it will die out. By understanding the 'turbulent kinetic energy budget' we can understand why the depth of the layer within which there is turbulence (causing momentum transfer, heat transfer etc.) undergoes diurnal and seasonal cycles[4].

In slightly simplified form the TKE budget for the horizontally-uniform ABL is

$$\begin{aligned}\frac{\partial k}{\partial t} + \overline{u}_j\,\frac{\partial k}{\partial x_j} = 0 \quad &= \quad -\overline{u'w'}\,\frac{\partial \overline{u}}{\partial z} - \overline{v'w'}\,\frac{\partial \overline{v}}{\partial z} + \frac{g}{T_0}\,\overline{w'T'} \\ &- \quad \epsilon\ + \text{(term often modelled as diffusion)}\end{aligned} \tag{6}$$

(by choice of our coordinate system it is possible in the *surface layer* to eliminate the term involving $\overline{v}$).

The first two term on the rhs are conventionally called "shear production". The stresses due to unresolved scales work on the mean flow, converting MKE (mean KE) to TKE[5]. The third term on the rhs is buoyant production, the mean rate of working (force x velocity) by the buoyancy force $\overline{(g\,T'/T_0)\;w'}$. The term represented by ϵ is the turbulent kinetic energy dissipation rate, and if written explicitly it involves viscosity. It is often modelled as $\epsilon = k/\tau$ where τ is a timescale. The terms indicated as 'modelled as diffusion' comprise a transport term, made up of two parts, pressure transport and turbulent transport.

Local Equilibrium

The term "local equilibrium" describes the situation when the transport term is negligible, so that local production balances local dissipation.

$$\frac{\partial k}{\partial t} = 0 = -\overline{u'w'}\,\frac{\partial \overline{u}}{\partial z} + \frac{g}{T_0}\overline{w'T'} - \epsilon \tag{7}$$

With the added restriction of neutrality ('hhNSL'), the TKE budget reduces to

$$\frac{\partial k}{\partial t} = 0 = -\overline{u'w'}\,\frac{\partial \overline{u}}{\partial z} - \epsilon \tag{8}$$

[4]The TKE budget also offers many other crucial insights into the fluid dynamics of turbulence.

[5]Recently McNaughton has given an alternative interpretation of "shear production" as being, in reality, the vertical divergence of a mean vertical flux of kinetic energy; at any level z, this flux from the atmosphere above z does work on the atmosphere below z to maintain the flow.

which is often written as:

$$P = -\overline{u'w'}\,\frac{\partial \overline{u}}{\partial z} = \epsilon \tag{9}$$

where P is the TKE production rate. Now we shall later see that in the hhNSL the wind shear is $u_*/k_v z$, so we have:

$$P = \epsilon = \frac{{u_*}^3}{k_v\ z} \tag{10}$$

It is often assumed that, not too close to the surface ($z \gg z_0$) the horizontally-uniform atmospheric surface layer is in local equilibrium.

Flux Richardson Number

The Flux Richardson number is defined by the ratio of the two energy source terms in the TKE budget:

$$R_i{}^f = \frac{\frac{g}{T_0}\ \overline{w'T'}}{\overline{u'w'}\ \frac{\partial \overline{u}}{\partial z}} \tag{11}$$

$R_i{}^f$ is negative under unstable stratification, zero for neutrality, and positive in stable stratification (when buoyancy forces suppress vertical motion and thus act as a sink for TKE). In unstable stratification it is found empirically that $R_i^f \sim z/L$, where L is the Obukhov length (see below).

The flux Richardson number is one of several common 'stability parameters' quantifying the effect of stratification on the hhASL. These can all be related to the fundamental stability parameter z/L of the Monin-Obukhov similarity theory.

Eddy size range & the TKE spectrum

A high Reynolds-number turbulent flow field (such as the ABL) consists of a large number of chaotic, self-distorting (and stretching) vortices. "The scale of a vortex which is stretched decreases - this is the mechanism for passing energy from larger to smaller scales." (Tennekes and Lumley, p75). In turbulent flows energy is generally fed from large vortices (generated by flow interaction with the surfaces, or by buoyancy) to smaller and smaller. "The interaction of vortices (mutual stretching and tilting) leads to a chaotic process (turbulence) in which energy is fed from large vortices to smaller and smaller vortices (the energy cascade) and finally dissipated by molecular viscosity."

"Kinetic energy passes from the mean flow (if a mean strain rate is present) down through vortex motions of smaller and smaller scale until it is converted into thermal internal energy by viscosity. This process is independent of viscosity except in the final stage. So the rate of energy transfer to the smallest motions is independent of viscosity. Viscosity causes dissipation but does not control its rate; the intensity and scale of the small scale motion adjust themselves so as to dissipate all the energy transferred from larger scales, and the smaller the viscosity the smaller the motions that can survive." (Bradshaw).

Recall that the power spectrum of (e.g.) the u velocity component $S_{uu}(f)$ [m^2 s^{-2} (s^{-1})$^{-1}$ $\equiv$ m^2 s^{-1}], illustrated schematically by Fig. 1, gives the range of frequencies (f) within which the variance in u occurs:

$$\sigma_u{}^2 = \int_0^\infty S_{uu}(f)\, df \tag{12}$$

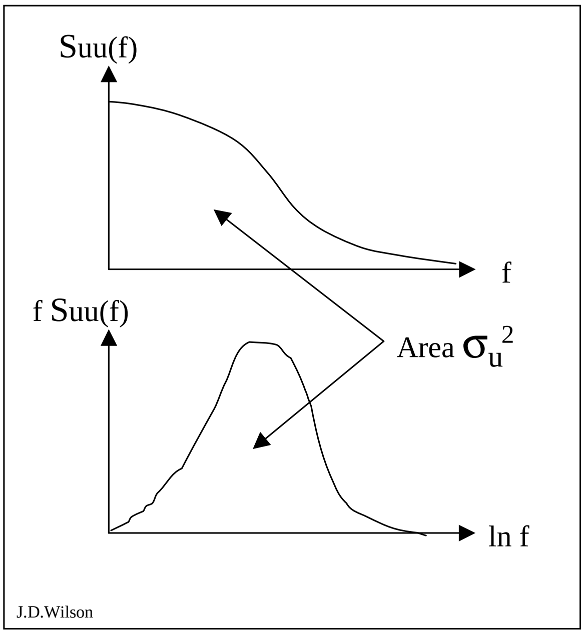

Figure 1: *Schematic of two common plot styles for the power spectrum of the fluctuation u'. In each case, the area under the curve gives the variance.*

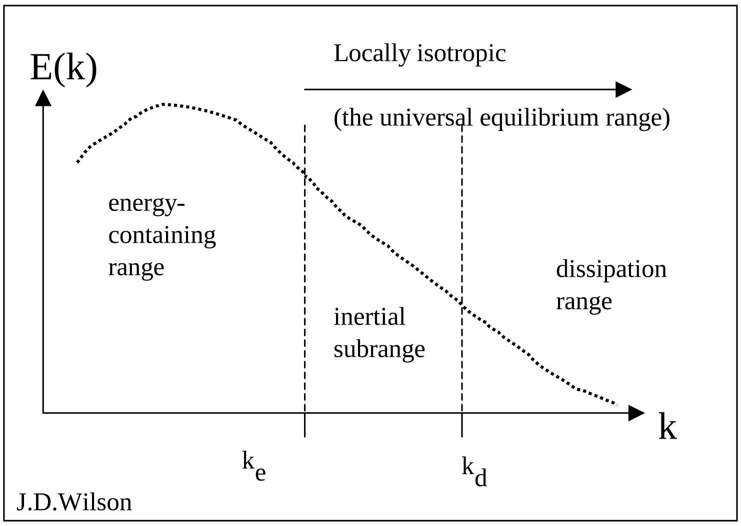

Figure 2: *Schematic wavenumber spectrum. If l_e is a length scale characterizing the largest eddies, which are likely to be on the scale of the flow domain (eg. ABL depth δ) then a wavenumber characteristic of the production region is $k_e \sim 1/l_e$. The wavenumber marking the transition into the dissipation range (where viscosity plays a role) is $k_d \sim 1/\eta_K$ where η_K is the Kolmogorov length*

Now consider Fig. (2), a schematic wavenumber spectrum for the turbulent kinetic energy defined such that

$$\mathrm{TKE} = \frac{1}{2}\left(\sigma_u^2 + \sigma_v^2 + \sigma_w^2\right) = \int_0^\infty E(k)\, dk \tag{13}$$

(its a bit unfortunate that I chose a wavenumber spectrum rather than a frequency spectrum, because we often use k for the TKE itself!).

The shape of the spectrum in the energy-containing range depends on the specifics of the flow. However the smaller eddies of the universal equilibrium range are considered to be isotropic and to have statistics independent of details of energy production, depending only on the overall spectral flux[6] ϵ and (in the dissipation range) the viscosity ν. As the Reynolds number increases,

[6] According to the notion of an energy "cascade", the transfer-rate down-scale is just ϵ, which is the rate of conversion of TKE to heat. That is, according to this idealized spectral equilibrium of the mix of eddies, TKE is not "accumulated" in any scale-range but rather, is produced at rate ϵ at large scales, dissipated at rate ϵ at dissipation scales, and in short, merely "passed down" the intermediate range of scales.

the separation in scale $k_d - k_e$ between the dissipation range and the energy-production range increases.

One of the many famous and useful results due to Kolmogorov is a dimensional analysis (see Appendix) determining the approximate size η_K of the the smallest turbulent eddies, which Kolmogorov postulated to be controlled by the kinematic viscosity ν and the TKE dissipation rate ϵ. From Buckingham's pi-theorem we have $n = 3$ variables in $m = 2$ dimensions (length + time) so the law we seek contains only a single dimensionless ratio:

$$\frac{\eta_K}{(\nu^3/\epsilon)^{1/4}} = c \tag{14}$$

where (by convention) $c = 1$.

Neutral wind profile for the surface layer via K-theory

What does the (putative - or rather, approximate) constancy of $\overline{u'w'}$ tell us about the mean wind, $\overline{u}(z)$? Nothing. So far, we only know that $\overline{u} = 0$ at ground; and we know it increases aloft.

In a laboratory wall shear layer the Coriolis force is negligible and consequently there is no turning of wind direction of height. A suitable empirical formula for the mean velocity as function of height is

$$\overline{u}(z) = U_\delta \left(\frac{z}{\delta}\right)^m \tag{15}$$

where U_δ is the free stream velocity (velocity outside the boundary layer) and m is an empirical parameter. Suitably tuned, this can be made to fit the mean windspeed or perhaps even mean velocity over a restricted region of the ABL as well, typically by taking a reference height H that might be of order of the surface layer depth z_S, viz.

$$\overline{u}(z) = U_H \left(\frac{z}{H}\right)^m \tag{16}$$

But this is pure empiricism, and not very physical. To obtain a more satisfying and insightful formula for $\overline{u}(z)$ for the neutrally-stratified and horizontally-uniform surface layer, we will turn to the closure assumption (which we shall tentatively assume useful), which relates the velocity *gradient* to the shear stress:

$$\overline{u'w'} = -K_m \left(\frac{\partial \overline{u}}{\partial z} + \frac{\partial \overline{w}}{\partial x} \right) \tag{17}$$

(correct symmetry demands the term involving $\overline{w}$, but we know it vanishes in our hhASL).

K-theory was introduced by the pioneers as a model for the turbulent convective fluxes, which they recognized as needing to be known but were unable to measure. K-theory represents turbulent convection as if it were a diffusive process, ie. models turbulent convection by analogy to molecular diffusion. It is therefore fundamentally wrong - but it works, in simple situations.

Now the eddy viscosity K_m [m^2 s^{-1}] is (dimensionally) the product of a velocity scale times a length scale. It has (qualitatively) to do with the efficiency of the turbulence (ie. the unresolved scales of motion) in causing down-gradient transfer of momentum. So why not choose σ_w as the velocity scale? No vertical motion - no mixing. Also, $\sigma_w \propto u_*$ and u_* relates to TKE; so we can pick either u_* or σ_w as our scale. It is conventional to pick u_*.

Observations have shown that the turbulence lengthscale[7] (however defined) increases linearly with distance from ground (this is only true quite close to ground: it does not increase indefinitely). This suggests using as our length scale a multiple of z, say $k_v z$ (where k_v is called von Karman's constant). Doing so, we have:

$$-{u_*}^2 = -k_v \; z \; u_* \frac{\partial \overline{u}}{\partial z} \tag{18}$$

or rearranging:

$$\frac{\partial \overline{u}}{\partial z} = \frac{u_*}{k_v \; z} \tag{19}$$

If we define a constant of integration z_0 (the "surface roughness length") as the height where $\overline{u} = 0$, we can integrate to get the "log wind profile"

$$\overline{u}(z) = \frac{u_*}{k_v} \; \ln \left(\frac{z}{z_0} \right) \tag{20}$$

Experiments show that with $k_v = 0.4$ this is a good description of the mean wind in the undisturbed surface-layer under neutral stratification, provided we are not too close to ground ($z \gg z_0$). This of course prohibits use of the log profile within and close above crops. We will

[7]This may be defined formally. For example one may take as a time scale the reciprocal of the frequency at which the power spectrum peaks, then multiply by the velocity standard deviation. The length scale is considered to characterize the size of the large, energetic eddies - not the isotropic small fry.

later see how the Monin-Obukhov similarity theory generalizes our finding for the stratified (but still horizontally-uniform) surface layer, without need to introduce K-theory.

By measuring the mean wind $\overline{u}$ at two (but preferably several) heights we can deduce the friction velocity from the slope:

$$\frac{\Delta \overline{u}}{\Delta \ln(z)} = \frac{u_*}{k_v} \tag{21}$$

(and we can obtain z_0 from the $\overline{u} = 0$ intercept). Thus, bearing in mind that once we know u_* we can infer many other properties of the surface layer (surface drag τ, turbulence TKE, σ_w, etc.), a very great deal of information can be obtained simply by measuring the mean wind profile.

Remember: the log law is severely limited in its validity.

Monin-Obukhov Similarity Theory (MOST)

Why do we need a similarity theory? Because the governing equations for the mean flow are unclosed, and even when closure assumptions are supplied, they cannot be solved except numerically.

The MO theory is very simple, and has proven very successful. Its use is now standard in many fields, including surface layer treatment in weather and climate models. Note though, that (strictly) MOST is valid only in *horizontally-uniform* conditions, and only at heights $z \gg z_0$.

Monin and Obukhov assumed there is a layer far enough above ground that scales of surface roughness do not affect the flow, yet close enough to ground that the depth δ of the ABL does not affect the flow. Within this layer $z_0 \ll z \ll \delta$ it is assumed that the turbulence is "controlled" by:

- the kinematic heat flux through the surface layer $\overline{w'T'} = Q_H/(\rho c_p)$
- the kinematic momentum flux through the surface layer, $\overline{u'w'} = -\,{u_*}^2$
- a buoyancy parameter g/T_0, where T_0 is the mean (Kelvin) temperature

From the dimensional vantage point (see Appendix), this suggests using these scales for non-dimensionalisation:

$$
\begin{aligned}
& u_* \\
T_* &= -\frac{Q_H}{\rho c_p \; u_*} \\
L &= \frac{-{u_*}^3}{k_v \; \frac{g}{T_0} \; \frac{Q_H}{\rho \; c_p}}
\end{aligned} \tag{22}
$$

(note that only two of the set u_*, T^*, L are independent). The von Karman constant k_v is included in definition of L simply by convention. Note that T_0 would have been an unhelpful choice for a temperature scale: we want to quantify the influence of stratification, ie. of a heat flux through the surface layer. T_0 contains no information on that flux: a surface layer having $T_0 = 300^o K$ could be stable, neutral, or unstable!

With this choice of scales, for the wind shear we might write:

$$
\frac{k_v z}{u_*} \; \frac{\partial \overline{u}}{\partial z} = \phi_m \left(\frac{z}{L} \right) \tag{23}
$$

where $\phi_m()$ is some unknown "universal function." Of course $k_v z$ might equally as well be replaced by L, in which case we should have a different function on the rhs, ϕ^*.

This probably seems arbitrary. A systematic derivation of Monin-Obukhov scaling is easy, but let's just say that MOST is justified by its utility. Its simple, and it works fairly well.

MO Surface-Layer Profiles

Now the purpose of the similarity theory is to find compact and useful formulae for the properties of the surface-layer: and properties of dominant interest are the profiles of the mean wind $\overline{u}$, temperature $\overline{T}$, and humidity ρ_v. Since scales pertaining to the ground have not been included, we will work with the gradients in these properties. We write:

$$
\begin{aligned}
\frac{k_v z}{u_*} \frac{\partial \overline{u}}{\partial z} &= \phi_m \left(\frac{z}{L} \right) \\
\frac{k_{vh} z}{T_*} \frac{\partial \overline{T}}{\partial z} &= \phi_h \left(\frac{z}{L} \right) \\
\frac{k_{vw} z}{\rho_{v*}} \frac{\partial \overline{\rho_v}}{\partial z} &= \phi_w \left(\frac{z}{L} \right) \\
\frac{k_{vc} z}{c_*} \frac{\partial \overline{c}}{\partial z} &= \phi_c \left(\frac{z}{L} \right)
\end{aligned} \tag{24}
$$

Here $k_v z$ (etc.) could with equal validity be replaced by z or by L. This merely changes the numerical values of the dimensionless Monin-Obukhov universal similarity functions which appear on the rhs. The argument of these functions is the dimensionless height z/L. We begin by intentionally distinguishing the various von Karman constants, ie. $k_v, k_{vh}, k_{vw}, k_{vc}$, in order to require that the universal functions $\phi_m, \phi_h, \phi_w, \phi_c$ take unit value in the neutral limit, $z/L \to 0$.

But there is one aspect of the above expressions that needs to be modified. The mean temperature profile that corresponds to the state of zero heat flux (ie. neutrality, $T_* = 0$) is not $\partial\overline{T}/\partial z = 0$ but

$$\frac{\partial \overline{T}}{\partial z} = \Gamma \tag{25}$$

where $\Gamma = \frac{g}{c_p} \approx 0.01$ [K m^{-1}] is the "adiabatic lapse rate". Alternatively, of the neutrally-stratified layer we may say that the mean potential temperature $\overline{\theta}$ is uniform,

$$\frac{\partial \overline{\theta}}{\partial z} = 0 \tag{26}$$

The ASL is sufficiently deep that we need to re-write our equation for the temperature profile as:

$$\frac{k_{vh} z}{T_*} \left(\frac{\partial \overline{T}}{\partial z} - \Gamma \right) = \phi_h \left(\frac{z}{L} \right) \tag{27}$$

or better as

$$\frac{k_{vh} z}{T_*} \frac{\partial \overline{\theta}}{\partial z} = \phi_h \left(\frac{z}{L} \right) \tag{28}$$

Now we have expressions for the gradients in wind, temperature, humidity. This doesn't get us far till we know the universal dimensionless (ϕ) functions. These have been the subject of a handful of major micrometeorological "flux-gradient" experiments (so called because they established the relationships between fluxes and gradients) in the past three decades, and are now well-established. Before looking at the functions, a word about the Monin-Obukhov length L, and about the concept of eddy diffusivity (which has not arisen so far in this similarity analysis).

About the Obukhov Length

Loosely, L is a "stability parameter," positive in stable stratification, negative in unstable stratification, and infinite (+ or -) under neutral stratification. L is sometimes called the "height of the substrate of dynamic turbulence," because its magnitude indicates (speaking qualitatively) the depth of the layer in which shear production $-\overline{u'w'}\ \partial \overline{u}/\partial z$ of TKE is more important in the TKE balance than buoyant production/destruction $g/T_0\ \overline{w'T'}$. The ratio of these terms in the TKE budget is by definition the flux Richardson number and the flux-gradient experiments have shown that (in the case of unstable stratification):

$$R_i{}^f = \frac{\frac{g}{T_0}\ \overline{w'T'}}{\overline{u'w'}\ \frac{\partial \overline{u}}{\partial z}} \approx \frac{z}{L} \tag{29}$$

Exchange coefficients

The MO similarity theory does not depend on the introduction of K-theory. However if we wish to introduce the eddy viscosity (eg.), we have by definition:

$$\overline{u'w'} = -K_m\ \frac{\partial \overline{u}}{\partial z} \tag{30}$$

and comparison with the above relation for the wind shear shows that we must have:

$$K_m = \frac{k_v u_* z}{\phi_m \left(\frac{z}{L}\right)} \tag{31}$$

Similarly,

$$\begin{aligned} K_h &= \frac{k_v u_* z}{\phi_h \left(\frac{z}{L}\right)} \\ K_w &= \frac{k_v u_* z}{\phi_w \left(\frac{z}{L}\right)} \end{aligned} \tag{32}$$

Thus we are supplied with closure relations, but these are valid only under the conditions such that MO similarity is valid: not too close to the "ground" or canopy, in a horizontally-uniform ASL.

Relationship between Flux and Gradient Richardson Numbers

We've already encountered the flux Richardson number, defined by eqn(29). Substituting the Monin-Obukhov expressions for the the wind shear it follows that:

$$R_i{}^f = \frac{z}{L}\ \frac{1}{\phi_m} \tag{33}$$

The gradient Richardson number is another dimensionless stability parameter, which can be evaluated from gradients alone:

$$R_i{}^g = \frac{\frac{g}{T_0}\frac{\partial \overline{T}}{\partial z}}{\left(\frac{\partial \overline{u}}{\partial z}\right)^2} \tag{34}$$

Using the MO expressions for the temperature and velocity gradients, this may be written:

$$R_i{}^g = \frac{z}{L}\frac{\phi_h}{\phi_m{}^2} = \frac{\phi_h}{\phi_m}\, R_i{}^f \tag{35}$$

Experimental data are mostly consistent with $\phi_h \approx \phi_m{}^2$.

The ITCE (International Turbulence Comparison Experiment)

Yaglom (1977) gives an excellent discussion of the difficulty of the experiments to determine the MO functions. The desire to determine these ϕ's spurred development of fast-response sensors for direct measurement of the fluxes (eddy correlation). In the days before computers, the need for fast sampling and cross multiplication of signals lead to some very ingenious electronic work, based on analog rather than digital processing (eg. a fascinating glimpse of the concepts and equipment of micro-meteorology in the 1950's is given by Halstead et al., 1957).

The ITCE experiment was lead by CSIRO, with participation by Argonne National Labs and a group from the Soviet Union. The site was flat, open grazing land in New South Wales, with the slope less than 2/10000. Fast response sensors determined all covariances (fluxes) and turbulence statistics, while at the ground drag plates determined drag (τ_0) and a Lysimeter the evaporation rate (E_0). In conjunction with the flux measurements, masts supported cup anemometers and temperature sensors.

The ITCE analysis by Dyer and Bradley (1982) did not presuppose the value of k_v, their starting point being

$$\begin{aligned}
\frac{k_{vm} z}{u_*}\,\frac{\partial \overline{u}}{\partial z} &= \phi_m\left(\frac{z}{L}\right)\\
\frac{k_{vh} z}{T_*}\,\frac{\partial \overline{T}}{\partial z} &= \phi_h\left(\frac{z}{L}\right)\\
\frac{k_{vw} z}{\rho_{v*}}\,\frac{\partial \overline{\rho}_v}{\partial z} &= \phi_w\left(\frac{z}{L}\right)
\end{aligned} \tag{36}$$

where $q_* = -E/\rho u_*$, $T_* = -Q_H/\rho c_p u_*$. As noted earlier, allowing different k_v's permits to insist that $\phi_m(0) = \phi_h(0) = \phi_w(0) = 1$ for the neutral limit $z/|L| = 0$. Dyer and Bradley concluded that within the accuracy of the experiment, $k_{vm} = k_{vh} = k_{vw} = 0.4$ (so we can call them all simply k_v) and therefore the eddy diffusivities are all equal in neutral stratification[8]. For the similarity functions in the range $-4 < z/L < -0.004$ of unstable stratification they recommend

$$\begin{aligned} \phi_m &= \left(1 - 28\,\frac{z}{L}\right)^{-\frac{1}{4}} \\ \phi_h = \phi_w &= \left(1 - 14\frac{z}{L}\right)^{-\frac{1}{2}} \end{aligned} \tag{37}$$

In stable stratification the result of Webb (1970) is usually used:

$$\phi_m = \phi_h = \phi_w = 1 + 5\frac{z}{L} \tag{38}$$

Profiles in stable stratification

If the ϕ functions for the stable side are written $\phi = 1 + \beta z/L$ (where $\beta \sim 5$) then the profiles are:

$$\begin{aligned} \overline{u}(z) &= \frac{u_*}{k_v}\left(\ln\,\frac{z}{z_0} + \beta\,\frac{z - z_0}{L}\right) \\ \overline{T}(z) - \overline{T}(z_{0T}) &= \frac{T_*}{k_v}\left(\ln\,\frac{z}{z_{0T}} + \beta\,\frac{z - z_{0T}}{L}\right) \end{aligned} \tag{39}$$

Since the diffusivities are all the same, all profiles have this log-linear form. In some contexts it is important to distinguish different roughness lengths for momentum and heat.

Velocity Standard Deviations

The “power in w,” measured by ${\sigma_w}^2$, is controlled by a number of counterbalancing influences (notably buoyant production/extraction, redistribution to/from other components, vertical transport, and viscous dissipation). There is no reason to expect σ_w to have a simple distribution in the ASL. It must surely vanish “at” ground. But despite the complexity of the mechanisms which set the level of ${\sigma_w}^2$, to a rough first approximation the MOST prediction that

$$\frac{\sigma_w}{u_*} = \phi_{ww}\left(\frac{z}{L}\right) \tag{40}$$

[8]But see a later section that casts some doubt on the generality of this finding.

does apply, except in the UBL (unresolved basal layer) adjacent to ground (where $z \sim O[z_0]$), and which might be a tall plant canopy. There is modern evidence that $\phi_{ww}()$ has other arguments.

Formulae such as (Panofsky et al., 1977)

$$\frac{\sigma_w}{u_*} = \left[1.6 + 2.9(-z/L)^{2/3}\right]^{1/2} \tag{41}$$

or (Kaimal and Finngan, 1994)

$$\frac{\sigma_w}{u_*} = 1.25 \left(1 - 3\frac{z}{L}\right)^{1/3} \tag{42}$$

are clearly to be taken as (at best) giving σ_w/u_* *in the mean* for given z/L; but even at an ideal site, values over any observation interval may differ from these formulae by a large amount, eg. neutral values need not cluster very closely about $\sigma_w/u_* = 1.3$, at least on the evidence of this classic paper.

In the case of the horizontal components, it has been known for a long time that strict MO similarity does not apply. For example Calder (1966) noted that "the classical form of (MOST) cannot be applied legitimately to the variances of the horizontal components of the wind velocity fluctuation", and that this "has recently been suspected on the basis of observational data".

Townsend was the first to speak of "inactive turbulence", large, quasi-horizontal eddies with which is associated very little vertical motion and very little vertical transfer of momentum. The energy in the "inactive turbulence" is not correlated with the friction velocity, ie. the MOST velocity scale, and the lengthscale of those motions goes with BL depth δ. Consequently MOST is incomplete, as far as it pertains to $\sigma_{u,v}$ and it has been found to be useful to invoke δ as a further scale. According to Panofsky et al., and many other articles, the horizontal velocity standard deviations σ_u, σ_v in the unstably-stratified surface-layer obey the formula:

$$\frac{\sigma_{u,v}}{u_*} = [12 + 0.5(\delta / -L)]^{1/3} \tag{43}$$

Notice the implication for the 'turbulence intensity'. Taking the neutral case for simplicity, we have

$$\frac{u'}{\overline{u}} \sim \frac{\sigma_u}{\overline{u}} \approx \frac{2.3\, u_*}{(u_*/k_v)\ln(z/z_0)} \approx \frac{1}{\ln(z/z_0)} \tag{44}$$

which is small for large z/z_0. Thus in many (but not all) contexts we can neglect the fluctuation u' relative to the mean streamwise velocity. This implies the x axis is a '1-way' axis in many problems, in particular in (many) dispersion problems: throw a handful of dust in the air and it blows downwind.

Turbulent Prandtl and Schmidt Numbers

Ratios of the von Karman constants give us the turbulent Prandtl and Schmidt (S_c) numbers:

$$\begin{aligned} \frac{k_v}{k_{vh}} &= P_r = \left(\frac{K_m}{K_h}\right)_{|L|=\infty} \\ \frac{k_v}{k_{vc}} &= S_c = \left(\frac{K_m}{K_c}\right)_{|L|=\infty} \end{aligned} \tag{45}$$

Atmospheric "flux-gradient" experiments are not unanimous on the values of the von Karman constants for momentum, heat and vapour. An alternative and indirect source of information upon them stems from atmospheric tracer experiments. Eulerian dispersion models whose form reduces (in the limit of undisturbed, horizontally-uniform flow) to

$$\overline{u}\,\frac{\partial \overline{c}}{\partial x} = \frac{\partial}{\partial z}\left(K_c(z)\,\frac{\partial \overline{c}}{\partial z}\right) \tag{46}$$

with $K \propto {\sigma_w}^2$ apparently require $K_c/K_m \approx 1.6$ (ie. $S_c \approx 0.63$) for optimal agreement with the observations.

And finally, a quick indication of what's above the surface layer

The diurnal cycle in the components of the surface energy budget[9]

$$Q^* = Q_H + Q_E + Q_G \tag{47}$$

(where $Q_E = L\,E$ is the latent heat flux density corresponding to the evaporation rate E) provides much insight into the diurnal evolution of the (fair-weather, horizontally-uniform) ABL. For the time being, assume a dry system ($Q_E = 0$) and neglect the soil heat flux Q_G. Then the surface heat flux $Q_H \approx Q^*$, with Q^* large and positive by day (say, $Q^* \sim +500[W\ m^{-2}]$) due

[9] We consider the vertical fluxes of energy through an imaginary reference plane just above the surface. A plane cannot store energy, so the net flux must vanish. Here Q^* is positive for a positive net flux of radiant energy directed downwards towards the ground-air interface; all other fluxes are defined as positive for energy flow away from the ground-air interface.

to strong net solar insolation, and moderately negative at night due to net longwave radiative loss ($Q^* \sim -100[W\ m^{-2}]$). Now, this strong surface heat flux implies daily dumping into the ABL of a substantial quantity of heat, with the converse at night, ie., a downward flow of heat to ground. Thus we expect instability (near the surface) by day and stability (temperature inversion) by night.

But recall that the buoyancy-production term $(g/T_0)\ \overline{w'T'}$ in the TKE equation enhances vertical exchange under unstable stratification (day), but damps vertical motion during stable stratification (by night). Hence we can expect the daytime ABL to be strongly mixed, and to mix the daily injection of heat though a deep layer (ABL depth δ or z_i growing large by day). However the suppression of turbulence at night by a growing surface inversion usually means that heat gained by the ground from the atmosphere is extracted from a shallow turbulent layer (ABL depth δ shrinking by night).

A natural feedback, easily visualized as the TKE production term $(g/T_0)\ \overline{w'T'}$, prevents the development of extreme instability in the atmosphere, except immediately adjacent to ground, where the short time-scale of w' fluctuations limits mixing efficiency despite a strongly unstable temperature gradient. That is, away from the surface, buoyant production leading to strong mixing (large TKE) transports heat away from the surface layer into the bulk of the ABL along a temperature gradient that, due to the strength of mixing, need be only slightly on the unstable side of well-mixed (adiabatic). Ideally, then, the daytime CBL is "well-mixed", ie., its potential temperature $\overline{\theta} = \overline{\theta}(t)$ is height-independent (except in a shallow surface layer; see Fig.3) and increases steadily throughout the day due to the heat injection.

The above reasoning suggests the depth of mixing in the night-time ABL will be defined by the limited depth across which shear production is sufficiently large to overcome the buoyant suppression of turbulence, and so allow fluctuating vertical motion to be sustained.

Above the surface layer, the eddies are rather large (one might take δ as their length scale) and velocity statistics scale with the convective velocity scale

$$w_* = \left(\frac{g}{T_0}\ (\overline{w'T'})_{gnd}\ \delta \right)^{1/3} \tag{48}$$

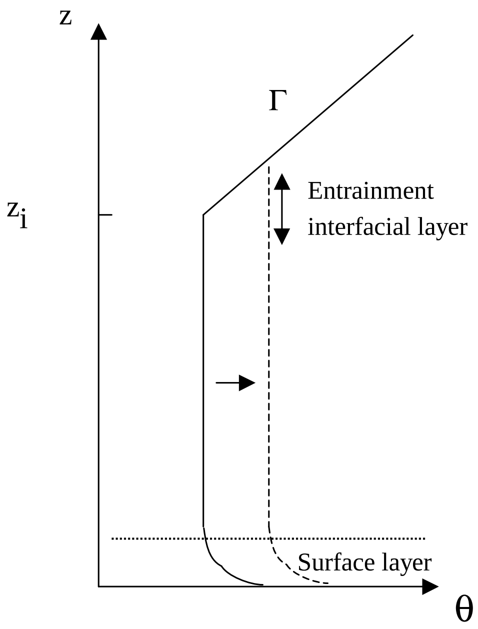

Figure 3: *Idealized profile of mean potential temperature $\theta(z)$ in the daytime CBL (ie.unstable ABL). Highly-unstable surface layer, well-mixed (near-adiabatic) outer layer, capping inversion (lapse-rate of potential temperature Γ). Height of the base of the inversion z_i grows as θ increases due to surface heat input.*

Appendix: Dimensional Analysis

Dimensional analysis is a tool of great power in the examination of any physical system. It can guide not only the theoretician, but also the design and analysis of experiments. According to Drazin & Reid (Hydrodynamic Stability, p12) the use of dimensional analysis in fluid mechanics was encouraged by Reynolds' discovery that stability (sustained plane-parallel, or laminar, motion) of Poiseuille flow in a pipe depends only on the dimensionless "Reynolds number" $R_e = Ud/\nu$ (U=velocity, d=diameter, ν = kinematic viscosity of air).

It is obvious, though we may never have remarked the fact, that physical laws must be dimensionally homogeneous. If $A = B+C$ is a meaningful law of a sensible physical system, then the dimensions $[A], [B], [C]$ of the terms must be equal: a valid equation cannot mix (additively) terms with units of kilograms and terms with units of seconds[10]. As a consequence of this

[10]This is sometimes called the "principle of dimensional homogeneity."

commonplace, the Buckingham Pi Theorem (named for Buckingham, 1914, but see White's "Fluid Mechanics" for independent discoverers) asserts that:

> "If an equation in n variables is dimensionally homogeneous with respect to
> m fundamental dimensions it can be expressed as a relation
> between n-m independent dimensionless groups."
> Buckingham 1914, Phys. Rev. 4, p345

Example: Pendulum

Suppose an observer suspects or hypothesizes that the period T of a simple pendulum is controlled by its mass M, length L, and gravity g (see Fig. 4).

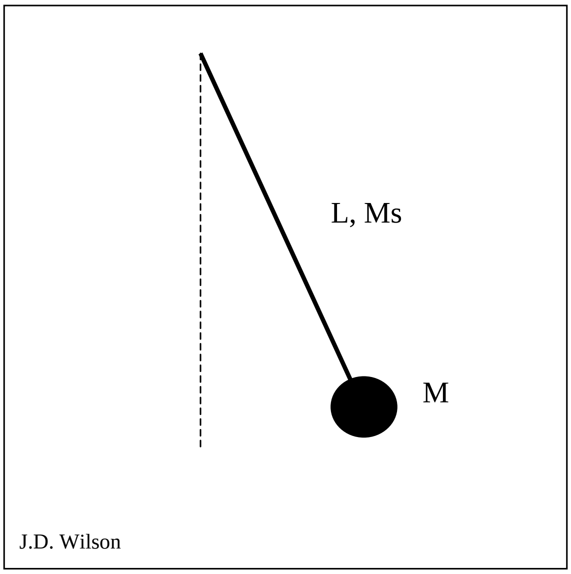

Figure 4: *The period (T) of a pendulum in gravity (g, ms^{-2}) is suspected to depend on the masses (M, M_s) and the length of the connection (L).*

The number of variables involved is $n = 4$ and we have $m = 3$ fundamental dimensions[11]. Thus the governing equation must involve only $n - m = 1$ non-dimensional variables.

[11] Time t has not been counted by this astute thinker, for he stipulates the process under investigation is a non-decaying, purely oscillatory motion, and does not ask to know any characteristic of the motion that *might* evolve in time.

Therefore s/he should seek an equation of form $N = const.$, where N is dimensionless. Only a single non-dimensional number can be formed from T, M, L, g, namely $\frac{T}{\sqrt{L/g}}$, so $\frac{T}{\sqrt{L/g}} = const.$

We say that the "controlling timescale" is (L/g). The validity of this prediction is confirmed by a force balance, which yields for small amplitude motion:

$$\frac{T}{\sqrt{L/g}} = 2\pi \tag{49}$$

The mass M has been found to be irrelevant. The inclusion of irrelevant variables adds work but does not invalidate the procedure. The omission of a truly relevant variable will yield a relationship that does not agree with reality.

If we add the possibility that the mass of the string M_s may be important, we now have $n = 5, m = 3$. The prediction is then:

$$\frac{T}{\sqrt{L/g}} = F(\frac{M}{M_s}) \tag{50}$$

and we know that $F \to 2\pi$ as $M/M_s \to \infty$.

References

Buckingham, E. 1914. On Physically Similar Systems: Illustrations of the Use of Dimensional Equations. *Phys. Rev.*, **4**, 345–376.

Calder, K.L. 1966. Concerning the Similarity Theory of A.S. Monin and A.M. Obukhov for the Turbulent Structure of the Thermally Stratified Surface Layer of the Atmosphere. *Quart. J. R. Meteorol. Soc.*, **92**, 141–146.

Dyer, A.J., & Bradley, E.F. 1982. An Alternative Analysis of Flux-Gradient Relationships at the 1976 ITCE. *Bound. Layer Meteorol.*, **22**, 3–19.

Garratt, J.R. 1992. *The Atmospheric Boundary Layer.* Cambridge University Press.

Halstead, M.H., Richman, R.L., Covey, W., & Merryman, J.D. 1957. A Preliminary Report on the Design of a Computer for Micrometeorology. *J. Meteorology*, **14**, 308–325.

Kaimal, J.C., & Finnigan, J.J. 1994. *Atmospheric Boundary Layer Flows.* Oxford University Press.

Panofsky, H.A., Tennekes, H., Lenschow, D.H., & Wyngaard, J.C. 1977. The Characteristics of Turbulent Velocity Components in the Surface Layer under Convective Conditions. *Bound. layer Meteorol.*, **11**, 355–361.

Webb, E.K. 1970. Profile Relationships: The Log-Linear Range and Extension to Strong Stability. *Quart. J. Royal Meteorol. Soc.*, **96**, 67–90.

Yaglom, A. M. 1977. Comments on Wind and Temperature Flux-Profile Relationships. *Bound. Layer Meteorol.*, **11**, 89–102.