

Exercise: calibrating the coefficients of a model for subgrid transport (EAS 471 March 27, 2014)

In many NWP models, the vertical fluxes carried by the unresolved scales of motion are modelled using an eddy viscosity, such that (for instance) the Reynolds stress $\overline{u'w'}$ would be modelled as

$$\overline{u'w'} = -K \left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} \right) \quad (1)$$

(where the term in $\partial W/\partial x$ vanishes in a horizontally-homogeneous flow). This closure of course requires that one adopt a scheme to specify K .

One of the classic approaches is to write

$$K = \lambda \sqrt{c_e k}, \quad (2)$$

where k is the “turbulent kinetic energy” of the unresolved velocity field, c_e is a dimensionless proportionality coefficient, and λ is a length scale. In such models a simplified transport equation for k (e.g. Eq. 7 below) is coupled with the momentum (and other) equations.

Now it is necessary to specify the length scale λ . In many instances this has been specified algebraically (for instance, we know that in a neutrally stratified and horizontally homogeneous surface layer, $\lambda = k_v z$). However to avoid that arbitrary imposition, for many flows it is common to adopt the “ $k - \epsilon$ ” model

$$K = c_\mu \frac{k^2}{\epsilon}, \quad (3)$$

whose implementation demands transport equations for both k and ϵ (again, c_μ is a dimensionless, constant coefficient).

A) Suppose we were to require that the eddy viscosity of Eq. (2) should be (algebraically) compatible with a “reference flow,” and let us take specifically the neutral surface layer as that reference flow. Then

$$U(z) = \frac{u_*}{k_v} \ln \left(\frac{z}{z_0} \right), \quad (4)$$

$$k = \frac{(c_u^2 + c_v^2 + c_w^2) u_*^2}{2} = \alpha u_*^2, \quad (5)$$

$$\lambda(z) = k_v z, \quad (6)$$

where U is the mean windspeed (assumed aligned along the x axis, such that $V = 0$), and the constants c_u (etc.) are normalized standard deviations of the velocity fluctuations ($c_w = \sigma_w/u_*$, etc). Along with $c_{u,v,w}$, the von Karman constant k_v , the friction velocity u_* , and the roughness length z_0 are all to be regarded as constants. Determine the value of the proportionality constant “ c_e ” in Eq. (2) that ensures exact satisfaction (in the given reference flow) of the steady-state turbulent kinetic energy balance equation:

$$\frac{\partial k}{\partial t} = 0 = K \left(\frac{\partial U}{\partial z} \right)^2 + \frac{\partial}{\partial z} \left(\frac{K}{\sigma_k} \frac{\partial k}{\partial z} \right) - \frac{(c_e k)^{3/2}}{\lambda}. \quad (7)$$

(Here σ_k is analogous to a turbulent Schmidt number or Prandtl number, for it merely adjusts an eddy diffusivity away from the eddy viscosity; it is usually assumed that $\sigma_k = 1$). What is the relationship between c_e and α ?

B) Given (further) that in the reference flow the profile of the TKE dissipation rate is (ideally)

$$\epsilon(z) = \frac{u_*^3}{k_\nu z}, \quad (8)$$

calibrate the coefficient c_μ of Eq. (3). Relate this to α and c_e (defined above).

C) Assuming steady state and horizontal homogeneity, the ϵ -equation associated with the standard $k - \epsilon$ model is

$$\frac{\partial \epsilon}{\partial t} = 0 = \frac{\epsilon}{k} (c_{\epsilon 1} P - c_{\epsilon 2} \epsilon) + \frac{\partial}{\partial z} \left(\frac{K}{\sigma_\epsilon} \frac{\partial \epsilon}{\partial z} \right). \quad (9)$$

Here P is the TKE production rate (sum of shear and buoyant production), which for the ideal reference flow defined above reduces to

$$P = K \left(\frac{\partial U}{\partial z} \right)^2 = \frac{u_*^3}{k_\nu z} \quad (10)$$

and so is equal to ϵ (the reference flow is said to be in “local equilibrium”).

Assuming profiles of U, k, ϵ appropriate to the reference flow, determine for what value of $\alpha (\equiv k/u_*^2)$ this ϵ -equation is exactly satisfied, if the following (which are the standard values) are taken for the coefficients:

$$c_\mu = 0.09, \quad (11)$$

$$\sigma_k = 1, \quad (12)$$

$$\sigma_\epsilon = 1.3, \quad (13)$$

$$c_{\epsilon 1} = 1.44, \quad (14)$$

$$c_{\epsilon 2} = 1.92. \quad (15)$$