EAS 471 Deriving a conservation eqn for "ϕ" 14 JAN/16

$\frac{D\phi}{Dt} = 0$? Only true if no process at "work" except advection

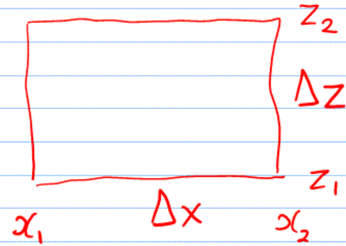
$\Rightarrow \frac{\partial \phi}{\partial t} = -\vec{u}.\nabla\phi$

But in general, many more processes are active.

We'll do a "budget" for ϕ on/in a control volume (in 2D for simplicity)

$[\phi]$ kg m^{-3} or number m^{-3} (amount per unit volume).

Let ϕ be average value in the c.v.



$\therefore$ total content $\phi \Delta x \Delta z$

How much does this change over interval Δt?

We need a symbol for rate of transport, say $\vec{F}$ [e.g. $kg\ m^{-2}\ s^{-1}$]

(the flux density of ϕ)

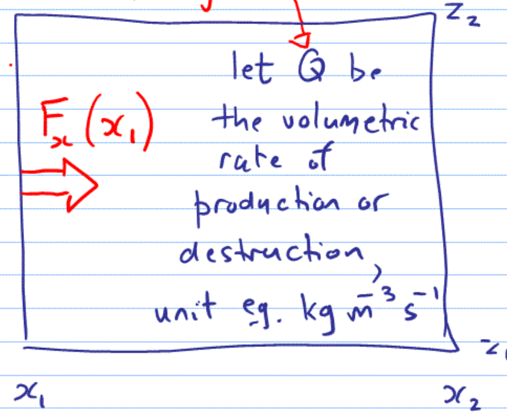
Interpret as the average along a face and over time step, thus $F_x(x_1)$ is the mean flux density across the x_1 face

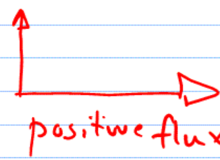
$\therefore$ $F_x(x_1) \times \Delta z \times \Delta t \times \Delta y$ [kg]

is the gain in the content of the c.v. during Δt by transport across face x_1 (can have either sign)

Gain by production is $Q\, \Delta x\, \Delta y\, \Delta z\, \Delta t$

interpret Q as an avg. throughout the c.v.





Change in total content of the c.v. during Δt is:

$$\Delta\phi\, \Delta x \Delta z \Delta y = \Delta y \Delta z \Delta t \left[F_x(x_1) - F_x(x_2) \right]$$

$$+ \Delta y \Delta x \Delta t \left[F_z(z_1) - F_z(z_2) \right]$$

$$+ Q \Delta x \Delta y \Delta z \Delta t$$

$\div$ by $\Delta x \Delta y \Delta z \Delta t$

$$\frac{\Delta\phi}{\Delta t} = \frac{F_x(x_2) - F_x(x_1)}{-\Delta x} + \frac{F_z(z_2) - F_z(z_1)}{-\Delta z} + Q$$

Shrink all "deltas" to zero

$$\frac{\partial\phi}{\partial t} = -\frac{\partial F_x}{\partial x} - \frac{\partial F_y}{\partial y} - \frac{\partial F_z}{\partial z} + Q$$

$$= -\nabla \cdot \vec{F} + Q$$

Let's specify $\vec{F}$ (now a point property, as are ϕ, Q)

$$\vec{F} = \vec{u}\phi - \rho \mathcal{D} \nabla \frac{\phi}{\rho} + \vec{R}$$

radiative flux (if any)

molecular diffusivity of ϕ in air $[m^2 s^{-1}]$

mixing ratio of ϕ in air

Fick's law of diffusion of species A in binary mixture A + "air"

Fourier law of conduction

$$\vec{Q}_H = -\rho c_p \kappa \nabla T$$

molecular thermal diffusivity $m^2 s^{-1}$

Let $\phi \rightarrow \rho_v$ (abs. humidity)

then $\vec{R} = 0$

(transport or "flux" form)

$$\frac{\partial \rho_v}{\partial t} = -\nabla \cdot \vec{u}\rho_v + \nabla \cdot \left(\rho \mathcal{D}_v \nabla q\right) + Q$$

$$= -\vec{u} \cdot \nabla \rho_v - \rho_v \nabla \cdot \vec{u} + \mathcal{D}_v \nabla \cdot \nabla \rho_v + Q$$

on ABL scale

advection form

$$\frac{\partial \rho_v}{\partial t} + \vec{u} \cdot \nabla \rho_v \equiv \frac{D\rho_v}{Dt} = -\rho_v \nabla \cdot \vec{u} + \mathcal{D}_v \nabla^2 \rho_v + Q \approx Q$$