EAS 471 Working towards a "heat equation" 19 JAN 2016

$$\phi \rightarrow \rho c_p T \quad \left[\frac{kg}{m^3} \frac{J}{kg\,K} K = J\,m^{-3} \right]$$

ρ air density, treat as const.

$c_p \simeq 1000 \; J\,kg^{-1}\,K^{-1}$

molecular conductivity

$$\rho c_p \frac{\partial T}{\partial t} = -\nabla \cdot \left(\vec{u}\, \rho c_p T - \rho c_p \kappa \nabla T + \vec{R} \right) + Q$$

molecular thermal diffusivity, $m^2 s^{-1}$

$$= -\rho c_p \nabla \cdot \vec{u}\, T + \rho c_p \kappa \nabla^2 T - \nabla \cdot \vec{R} + Q$$

neglect neglect

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T = -T\, \nabla \cdot \vec{u} + \kappa \nabla^2 T$$

neglect in ABL

Advection - Diffusion eqn

$$\frac{\partial \phi}{\partial t} = -\nabla \cdot \vec{F} + Q$$

Discretization using finite differences

$$\phi(x+\Delta x) = \phi(x) + \left(\frac{\partial \phi}{\partial x}\right)_x \Delta x + \frac{1}{2!}\left(\frac{\partial^2 \phi}{\partial x^2}\right)_x \Delta x^2 + \dots \quad \text{— I}$$

$$\phi(x-\Delta x) = \quad - \quad + \quad - \quad \text{— II}$$

I $\Rightarrow$

$$\left(\frac{\partial \phi}{\partial x}\right)_x = \frac{\phi(x+\Delta x) - \phi(x)}{\Delta x} + O(\Delta x)$$

forward difference

II

$$\left(\frac{\partial \phi}{\partial x}\right)_x = \frac{\phi(x) - \phi(x-\Delta x)}{\Delta x} + O(\Delta x)$$

I − II $\Rightarrow$

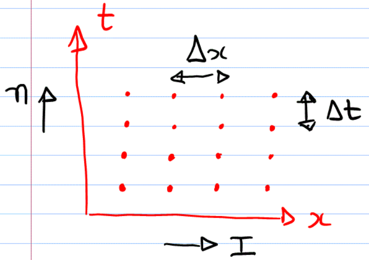
$$\left(\frac{\partial \phi}{\partial x}\right)_x = \frac{\phi(x+\Delta x) - \phi(x-\Delta x)}{2\Delta x} + O(\Delta x^2)$$

central difference

I + II

$$\left(\frac{\partial^2 \phi}{\partial x^2}\right)_x = \frac{\phi(x+\Delta x) + \phi(x-\Delta x) - 2\phi(x)}{\Delta x^2} + O(\Delta x^2)$$

Let's discretize the 1D "heat equation" $\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$



$$\frac{T_I^{n+1} - T_I^n}{\Delta t} = \kappa \frac{T_{I+1}^n + T_{I-1}^n - 2T_I^n}{\Delta x^2}$$

forward in time — central in space

Centred around time $n+\frac{1}{2}$ — Centred on time n

Truncation error = difference between pde and the difference eqn used to represent it.

$\underline{\underline{\text{this case}}}$ $O(\Delta t) + O(\Delta x^2)$

More terminology: a difference eqn is said to be consistent (compatible) with the pde it represents if the truncation error vanishes as the grid intervals are reduced indefinitely in size

Let ϕ = exact soln to a pde (typically unknown)

ϕ^* " " " difference eqn representing the pde

ϕ^{Num} = numerical solution to the difference eqn.

Discretization error $= \phi - \phi^*$

Stability error $= \phi^* - \phi^{Num}$

} sum is total error (TE)

$* \; TE = \phi - \phi^{Num} \xrightarrow[0]{\text{gridlengths}} 0$

Lax Equivalence Theorem: If a difference eqn is consistent with the pde it represents, then stability is the necessary and sufficient condition for convergence *