EAS 471 1^{st} Scored Assignment Due: Tues. 23 Feb. 2016

Eulerian Simulation of Dispersion in the ASL

Write a program to calculate the mean concentration field $\overline{c} = \overline{c}(x, z)$ downwind from a continuous crosswind line source[1] at $x = 0, z = h_s = 0.46$ m in the horizontally-uniform, neutrally-stratified atmospheric surface layer. Assume $\overline{c}$ is the solution of the advection-diffusion equation

$$\overline{u}(z)\,\frac{\partial \overline{c}}{\partial x} = \frac{\partial}{\partial z}\left[K(z)\,\frac{\partial \overline{c}}{\partial z}\right] \tag{1}$$

with mean windspeed and eddy diffusivity profiles

$$\overline{u}(z) = \frac{u_*}{k_v}\,\ln\frac{z}{z_0}\,, \tag{2}$$

$$K(z) = \frac{k_v}{S_c}\,u_* z \tag{3}$$

($k_v = 0.4$ is the von Karman constant, u_* is the friction velocity, z_0 is the surface roughness length and S_c is the turbulent Schmidt number; the x-axis points along the mean wind direction, and z, as usual, is the vertical axis).

Let J=1...N label a set of gridplanes along the vertical axis, and I=1... a set of gridplanes spaced along x. For this assignment use the following implicit discretization, derived in class using the control volume method:

$$\begin{aligned}\Delta z\,\overline{u}_J\,\left[\overline{c}_{I,J} - \overline{c}_{I-1,J}\right] &= \Delta x\,K_{J+\frac{1}{2}}\,\frac{\overline{c}_{I,J+1} - \overline{c}_{I,J}}{\Delta z}\\ &- \Delta x\,K_{J-\frac{1}{2}}\,\frac{\overline{c}_{I,J} - \overline{c}_{I,J-1}}{\Delta z}\,.\end{aligned} \tag{4}$$

This is a "marching" problem in the sense that the x (alongwind) axis is a "1-way" axis — there is no mechanism for material to travel against the wind. Therefore given the profile $\overline{c}_{I-1,J}$ for all J at location I-1, we can compute the profile at I by rearranging to obtain a set of neighbour equations linking $\overline{c}_{I,J}$ at all J (this discretization results in an implicit algorithm). The notation $J + \frac{1}{2}$ indicates that the height-varying eddy diffusivity $K(z)$ is to be evaluated on the interface separating the (J+1)th and Jth grid planes.

Specify grid-lengths $\Delta x \sim 0.5$ m, $\Delta z \sim 0.2$ m. For two choices $S_c = (1, 0.63)$ of the Schmidt number, compare your calculated solution $\overline{c}(100, z)$ at $x = 100$ m *on a graph* that also shows the observations (Table 1) of Project Prairie Grass run 57, for which the meteorological situation was $u_* = 0.50$ m s^{-1}, $z_0 = 0.0058$ m. (Note: to compare with Table 1, you'll need to scale your computed concentrations the same way, that is, you multiply your solution for $\overline{c}$ by $z_0 u_*/k_v$.)

[1] The field of $\overline{c}$ is the analog of the crosswind integrated concentration $\chi = \chi(x, z)$ due to a steady point source, and Project Prairie Grass provided field measurements of the latter.

Method

The above algorithm can be cast in the form

$$c_J\,\bar{c}_{I,J+1} + \; b_J\,\bar{c}_{I,J} + \; a_J\,\bar{c}_{I,J-1} = \; D_{I,J}, \quad \mathrm{J} = 1..\mathrm{J}_{\mathrm{mx}} \tag{5}$$

where the c_J, b_J, a_J are the "neighbour coefficients" (following the naming scheme of our Matlab solver), and $\bar{c}_{I,J}$ is the concentration matrix. The term $D_{I,J}$ contains information (only) from the upstream column at $I-1$, so we can regard it as a known. Thus we may frame our problem of finding the vertical column of values of $\bar{c}$ at downstream location I (given the column at $I-1$) as a matrix problem,

$$\mathbf{M}\,\mathbf{C} = \mathbf{D} \tag{6}$$

where $\mathbf{M}$ is the coefficient matrix, and is tridiagonal. Thus we may find the unknown column (i.e. $\bar{c}_{I,J}\ \forall J$) as

$$\mathbf{C} = \mathbf{M}^{-1}\,\mathbf{D} \tag{7}$$

which in Matlab syntax is $\mathbf{C} = \mathbf{M} \setminus \mathbf{D}$.

In principle we need only once compute $\mathbf{M}^{-1}$, and once known we could use it repeatedly to step down the I axis: each time we get a new $\mathbf{C}$ column matrix we recompute $\mathbf{D}$, and repeat the operation. Each step gives us $\mathbf{C}$ at a column further down the x-axis by a distance Δx.

To compute the new $\mathbf{C}$ column matrix for each step down the x-axis you may use either Matlab's built-in matrix inversion $\mathbf{C} = \mathbf{M} \setminus \mathbf{D}$, or a Matlab implementation of the (less robust) TDMA method ("TDMA_solver.m").

Upper and lower boundary conditions

Set the top of your domain $z(J_{max})$ sufficiently high (say, at least 30 m) that $\bar{c}(J_{max}) = 0$. Then your coefficients at $J = J_{max}$ are

$$\begin{aligned} b_{J_{max}} &= 1 \\ c_{J_{max}} &= \text{not used} \\ a_{J_{max}} &= 0 \\ D_{J_{max}} &= 0 \end{aligned} \tag{8}$$

If we presume our gas does not react with the ground, we want zero flux to ground, which is assured by requiring $\bar{c}(I,1) \equiv \bar{c}(I,2)$. Thus at $J = 1$ the needed coefficients are

$$\begin{aligned} b_1 &= 1 \\ c_1 &= -1 \\ a_1 &= \text{not used} \\ D_1 &= 0 \end{aligned} \tag{9}$$

Inlet boundary condition

How is your solution going to "know" there is a source? Let J_h be the height index of the cell the physical source will lie within, and let the streamwise index value $I = 1$ correspond to a column of gridpoints aligned at the source location. The easiest approach is to set the inlet or inflow concentration profile as

$$\overline{c}(1, J) = \begin{cases} 0 & \text{if } J \neq J_h, \\ 1/\left(\overline{u}(J_h)\ \Delta z\right) & \text{if } J = J_h \end{cases} \tag{10}$$

which guarantees that the total mass flux across the first interior plane $I = 1$ will be

$$Q = \sum_J \overline{c}(0, J)\ \overline{u}(J)\ \Delta z = 1 \tag{11}$$

Table 1: Normalized (and dimensionless) crosswind-integrated concentration $z_0 u_* \chi / (k_v Q)$ observed at distance $x = 100$ m from the point source (height $h_s = 0.46$ m) in Project Prairie Grass run 57; units are χ, $\mathrm{kg\,m^{-2}}$ and Q, $[\mathrm{kg\,s^{-1}}]$. (This is equivalent to the normalized concentration $z_0 u_* \overline{c}/(k_v\, q_\ell)$ at 100 m downwind from a crosswind line source at h_s, with $\overline{c}$, $[\mathrm{kg\,m^{-3}}]$ and Q, $[\mathrm{kg\,m^{-1}\,s^{-1}}]$.)

z [m]	$z_0 u_* \chi / (k_v Q)$
17.5	1.5E-6
13.5	6.6E-6
10.5	1.56E-5
7.5	3.51E-5
4.5	7.9E-5
2.5	1.25E-4
1.5	1.53E-4
1.0	1.62E-4
0.5	1.70E-4