EAS471 Mid-term Exam 11 Feb., 2016

Professor: J.D. Wilson Time available: 75 mins Value: 15%

Please answer four questions: 4 x 4% → 16%

1. The principle of "conservation of mass" can be expressed quite generally by the equation

$$\frac{\partial \phi}{\partial t} = -\nabla \cdot \mathbf{F} + Q \; , \tag{1}$$

where ϕ [$\mathrm{kg\,m^{-3}}$] is the volumetric concentration of the species of interest. Identify the assumptions or restrictions or simplifications whose introduction leads from Eq. (1) to the "diffusion equation"

$$\frac{\partial \phi}{\partial t} = \mathcal{D}_\phi \frac{\partial^2 \phi}{\partial x^2} \; , \tag{2}$$

where $\mathcal{D}_\phi$ is the molecular diffusivity of the species ϕ.

This is similar to short answer question 2 of the 2007 exam. Answer:

- assume there are no sources or sinks, i.e. $Q = 0$
- assume there is only one space dimension**, x, such that

$$\frac{\partial \phi}{\partial t} = -\frac{\partial F_x}{\partial x}$$

**or (equivalently), assume that symmetry prevails along other spatial axes such that $\partial F_y/\partial y = \partial F_z/\partial z = 0$.

- assume the flux of ϕ along x is purely diffusive, i.e. $F_x = -\mathcal{D}_\phi \, \partial\phi/\partial x$, such that

$$\frac{\partial \phi}{\partial t} = -\frac{\partial}{\partial x}\left(-\mathcal{D}_\phi \frac{\partial \phi}{\partial x}\right)$$

- assume the diffusivity is independent of position, then

$$\frac{\partial \phi}{\partial t} = \mathcal{D}_\phi \frac{\partial^2 \phi}{\partial x^2}$$

2. Perform a dimensional analysis to find a law for the terminal velocity w of a spherical particle of radius r and density ρ in still air with density ρ_a and kinematic viscosity $\nu\,[\mathrm{m^2\,s^{-1}}]$. The particle is subject to a gravitational force (per unit mass) given by $g\,[\mathrm{m\,s^{-2}}]$.

Answer: The number n of variables involved is $n = 6$, while the number of fundamental dimensions is $m = 3$. Therefore one seeks a relationship between $n - m = 3$ independent dimensionless variables, i.e. an expression of form

$$\pi_1 = f(\pi_2, \pi_3)$$

where π_1 will be the ratio of the terminal velocity w to some suitable scale for velocity such as, say, $\sqrt{rg}$. As to the arguments (π_2, π_3) of the unknown function, one of them can be chosen as ρ/ρ_a (or its reciprocal; either is equally acceptable). So we have

$$\frac{w}{\sqrt{rg}} = F\left(\frac{\rho}{\rho_a}, \theta\right)$$

where θ is the third, and as yet unspecified, dimensionless variable. How to specify the latter? We haven't yet "used" the kinematic viscosity ν, so any ratio we form using ν will be independent of the two we've already chosen. How to make ν dimensionless? Notice that the units of $\sqrt{r^3 g}$ are $\mathrm{m^2\,s^{-1}}$, the same as those of ν. Thus, finally,

$$\frac{w}{\sqrt{rg}} = F\left(\frac{\rho}{\rho_a}, \frac{\nu}{\sqrt{r^3 g}}\right) .$$

(Note: this is a simplified version of short answer question 2 of the 2009 exam, repeated as short answer question 1 of the 2010 exam. Notice that you did not need to do a laborious manipulation using the method of indices; however if you did choose to do so, carried through correctly it would have given the correct result.)

3. Determine the 4×4 tridiagonal coefficient matrix $\mathbf{M}$ and the right hand side $\mathbf{B}$ in a matrix expression of form $\mathbf{M}\ \boldsymbol{\Theta} = \mathbf{B}$ for the numerical solution of

$$\frac{\partial^2 \theta}{\partial z^2} = 2$$

on $0 \le z \le 1$, subject to $\theta(0) = \theta(1) = 0$. Set up your solution with four, equi-spaced gridpoints indexed $\mathrm{J} = (1, 2, 3, 4)$ positioned at $z_{\mathrm{J}} = (0, 1/3, 2/3, 1)$. Adopt the simple $\mathrm{O}[\Delta z^2]$ computational molecule for the curvature at internal gridpoints ($\mathrm{J} = 2, 3$). You need not invert $\mathbf{M}$, nor obtain the (numerical) solution vector $\boldsymbol{\Theta} = (\theta_1, \theta_2, \theta_3, \theta_4)$.

Answer: You have the following four equations:

$$\begin{aligned} 1 \times \theta_1 + 0 \times \theta_2 + 0 \times \theta_3 + 0 \times \theta_4 &= 0 \\ 9 \times \theta_1 - 18 \times \theta_2 + 9 \times \theta_3 + 0 \times \theta_4 &= 2 \\ 0 \times \theta_1 + 9 \times \theta_2 - 18 \times \theta_3 + 9 \times \theta_4 &= 2 \\ 0 \times \theta_1 + 0 \times \theta_2 + 0 \times \theta_3 + 1 \times \theta_4 &= 0\,. \end{aligned}$$

Thus,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 9 & -18 & 9 & 0 \\ 0 & 9 & -18 & 9 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,,\ \mathbf{B} = \begin{pmatrix} 0 \\ 2 \\ 2 \\ 0 \end{pmatrix}$$

(Note: this is a simplified version of short answer question 2 of the 2009 and 2010 exams).

4. Suppose a velocity "field" $U = U(x, t)$ is self-advecting along the x axis, i.e. $\partial U/\partial t = -U\,\partial U/\partial x$, and that initially the field is a pure sine wave with wavelength λ, i.e. the initial condition is $U(x, 0) = \sin kx$ where $k = 2\pi/\lambda$ is the wavenumber. In this context, and given the identities

$$\begin{aligned} \sin(A + B) &= \sin(A)\cos(B) + \cos(A)\sin(B)\,, \\ \sin(A - B) &= \sin(A)\cos(B) - \cos(A)\sin(B)\,, \end{aligned}$$

explain what is meant by "wave-wave interaction".

Answer: At $t=0$, we have $\partial U/\partial x = k\cos kx$ and so $\partial U/\partial t = -k\,\sin(kx)\cos(kx)$. It follows that an infinitesimal time dt later,

$$U(x,dt) = \sin(kx) - k\,dt\ \sin(kx)\cos(kx) = \sin(kx) - \frac{k\,dt}{2}\sin(2kx)$$

because $\sin(A)\cos(A) \equiv \frac{1}{2}\sin(2A)$. Wave-wave interaction (wavenumber k_1 with wavenumber k_2, though in this case $k_1 = k_2 = k$) produces waves of wavenumber (k_1+k_2) and (k_1-k_2).

5. Figure (1) shows a grid for the numerical solution of

$$\frac{\partial\theta}{\partial t} + U\,\frac{\partial\theta}{\partial x} = 0\ , \tag{3}$$

discretized as (say)

$$\frac{\phi_I^{n+1}-\phi_I^n}{\Delta t} + U\,\frac{\phi_I^n-\phi_{I-1}^n}{\Delta x} \tag{4}$$

and with finite gridlengths $\Delta x =$ const., $\Delta t =$ const. The sloping lines[1] correspond to two different values of the advecting velocity U, assumed constant. Explain for which of the two velocities this setup would satisfy the CFL condition $U\Delta t/\Delta x \le 1$, and why failure to do so would imply that the algorithm is unlikely to result in a satisfactory solution.

Answer: With the given algorithm, a "half-cone of influence" on the solution at (I,n) spreads outward along the x-axis towards smaller x by exactly the amount Δx for each increment Δt backward in time. We also know that the true value of $\theta(I,n) \equiv \theta(I\Delta x, n\Delta t)$ is $\theta(I\Delta x - Un\Delta t, 0)$, i.e. we look backward in time down the "ray" representing the physical velocity U and locate where it lies on the x axis at time $t=0$. In the case that $U = U_1$, this ray *lies outside the half-cone of influence* so that $\theta^{\rm Num}(I\Delta x, n\Delta t)$ cannot be influenced by the initial value at the proper point: the numerical solution must fail. Furthermore it is evident that $U_1\Delta t > \Delta x$, violating the CFL condition.

[1] In the actual exam, line U_2 was wrongly drawn as a negative velocity, such that as time increased position x decreased.

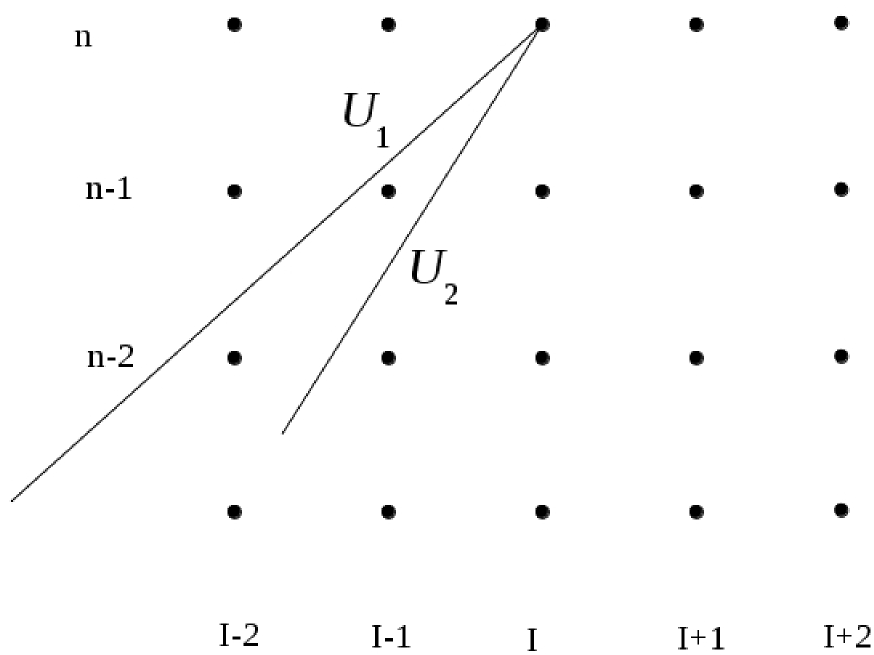

Figure 1: A uniform grid. Vertical axis is time, indexed n; horizontal axis is distance x, indexed I.