EAS 471 3^{rd} Scored Assignment (20%) Due: 7 Apr. 2016

Option B: Numerical simulation of an idealized windbreak flow

Suppose an infinitely long, porous windbreak of height H runs along the y-axis at $x = 0$, while the mean wind is oriented along the x-axis. We will suppose the surface layer is neutrally stratified, such that the profiles of mean wind speed and of eddy viscosity far upstream from the disturbance are

$$\overline{u}_0(z) = \frac{u_{*0}}{k_v} \ln \frac{z}{z_0} , \tag{1}$$

$$K_0(z) = k_v \, u_{*0} \, z \tag{2}$$

(u_{*0} is the upwind friction velocity, $k_v = 0.4$ is the von Karman constant, and z_0 is the surface roughness length; the subscript "0" denotes a property in the upwind, undisturbed region of the flow). We will further assume — however unrealistic this might appear[1]— that the disturbance to the flow does not alter the eddy diffusivity, i.e. at any point the local eddy viscosity $K(x,z) \equiv K_0(z)$. (For the purposes of the exercise, specific values for u_{*0}, z_0 and H are given below.)

The windbreak drags on the flow, and a plausible (and widely accepted) statement of momentum conservation reads:

$$\overline{u}\,\frac{\partial \overline{u}}{\partial x} + \overline{w}\,\frac{\partial \overline{u}}{\partial z} = \frac{-1}{\rho_0}\,\frac{\partial \overline{p}}{\partial x} - \frac{\partial\,\overline{u'^2}}{\partial x} - \frac{\partial\,\overline{u'w'}}{\partial z} - k_r\,\overline{u}\,|\overline{u}|\;\delta(x-0)\;s(z-H)\;. \tag{3}$$

This is the steady-state, Reynolds-averaged $\overline{u}$-mtm equation, the final term being an empirical "momentum sink", localized at the windbreak: k_r is known as the "resistance coefficient" or "pressure loss coefficient", and is dimensionless. The localizing functions are the delta-function $\delta(x-0)$ and a unit step function $s(z-H)$ defined

$$s(z-H) = \begin{cases} 1\,, & z \le H \\ 0\,, & z > H \end{cases} \;. \tag{4}$$

Notice that the product $\overline{u}\,|\overline{u}|$ guarantees that the drag, proportional in magnitude to the square of the wind speed, always *opposes* the wind.

[1]In fact it is not a bad first approximation. Wilson (1985) tested a range of "closure assumptions", including this one, i.e. an eddy viscosity closure using the undisturbed eddy viscosity $K_0(z)$. Simulations were compared against measurements, and this closure does capture the gross effect of the windbreak.

Numerical formulation

For our assignment we shall radically simplify Eqn. (3). Assuming the windbreak is very porous ($k_r \ll 1$) so that the disturbance to windspeed is small, we invoke a perturbation expansion

$$\begin{aligned}\overline{u}(x,z) &= \overline{u}_0(z) + k_r\,\overline{u}_1(x,z)\,, &(5)\\ \overline{w}(x,z) &= k_r\,\overline{u}_1(x,z)\,, &(6)\\ \overline{p}(x,z) &= k_r\,\overline{p}_1(x,z)\,. &(7)\end{aligned}$$

Then, retaining only terms to first order in k_r, one obtains

$$\overline{u}_0(z)\,\frac{\partial \overline{u}_1}{\partial x} = \frac{\partial}{\partial z}\left(K_0(z)\,\frac{\partial \overline{u}_1}{\partial z}\right) - \frac{1}{\rho_0}\,\frac{\partial \overline{p}_1}{\partial x} - \overline{u}\,|\overline{u}|\;\delta(x-0)\;s(z-H)\,. \tag{8}$$

where the shear stress has been modelled using an (undisturbed) eddy viscosity $K_0(z)$.

The sum of the pressure gradient and the momentum sink can be represented by the following *effective* pressure field (Wilson et al. 1990),

$$\frac{\overline{p}_1}{\rho_0}(x,z) = \frac{-\,\overline{u}_{0H}^2}{2\pi}\left[\tan^{-1}\left(\frac{H+z}{x}\right) + \tan^{-1}\left(\frac{H-z}{x}\right)\right] \tag{9}$$

in which $\overline{u}_{0H}$ is the windspeed at H far upstream from the windbreak. Eq. (8), a linear PDE, combines elements of equations studied earlier in the course[2]. At the inflow boundary the wind speed perturbation is zero, but it is "driven" away from zero by the effective pressure disturbance; and it is subject to streamwise advection and vertical diffusion (with a height-varying diffusivity). By taking a finite difference along the x-axis, Eq. (9) can be used to evaluate the pressure gradient wherever needed; however **for any cell that spans the barrier one must set $\partial \overline{p}_1/\partial x = 0$.**

Discretization

Let I,J be the horizontal and vertical indices for a uniform grid, with the spacing between nodes being $\Delta x, \Delta z$. Define your $z(\mathrm{J})$ as

$$z(\mathrm{J}) = z_0 + (\mathrm{J}-1)\,\Delta z \tag{10}$$

[2]In particular, $\overline{u}_1$ plays a role similar to the mean concentration $\overline{c}$ of the first scored assignment: it is advected by the same mean wind, and subject to vertical diffusion with an eddy diffusivity that differs from that we applied to concentration only by a constant factor S_c.

and your $x(\mathrm{I})$ as

$$x(\mathrm{I}) = X_0 + (\mathrm{I}-1)\,\Delta x \tag{11}$$

so that J = 1 corresponds to $z = z_0$ (where $\overline{u}_1$ vanishes) and I = 1 corresponds to $x = X_0$, the inflow boundary where, again, $\overline{u}_1$ vanishes. If $\mathrm{J_{mx}}$ indexes the highest row of gridpoints, then $\overline{u}_1(\mathrm{I},\mathrm{J_{mx}}) = 0$. Stated verbally, with this setup the wind disturbance vanishes on the inflow boundary, and along the uppermost and lowermost gridplanes. As x is a one-way axis, a downwind boundary condition is not needed, i.e. this is a marching problem on the x-axis.

Application of the control volume method (integration of the equation throughout a control volume surrounding the node) gives the following discretization, valid at all *interior* gridpoints ($1 < \mathrm{I}$ and $1 < \mathrm{J} < \mathrm{J_{mx}}$):

$$\begin{aligned}
\Delta z\,\overline{u}_0(\mathrm{J})\;[\overline{u}_1(\mathrm{I},\mathrm{J}) - \overline{u}_1(\mathrm{I}-1,\mathrm{J})] \;&=\; \Delta x\,K_{\mathrm{J}+1/2}\,\frac{\overline{u}_1(\mathrm{I},\mathrm{J}+1) - \overline{u}_1(\mathrm{I},\mathrm{J})}{\Delta z} \\
&-\; \Delta x\,K_{\mathrm{J}-1/2}\,\frac{\overline{u}_1(\mathrm{I},\mathrm{J}) - \overline{u}_1(\mathrm{I},\mathrm{J}-1)}{\Delta z} \\
&-\; \mathcal{S}_\mathrm{I}\,\Delta z\,\left[\frac{\overline{p}_1}{\rho_0}(\mathrm{I}+1/2,\mathrm{J}) \;-\; \frac{\overline{p}_1}{\rho_0}(\mathrm{I}-1/2,\mathrm{J})\right]\;,
\end{aligned} \tag{12}$$

where the "switch" $\mathcal{S}_\mathrm{I}$ is defined

$$\mathcal{S}_\mathrm{I} = \begin{cases} 1 & \text{if } x(\mathrm{I}) \neq 0\,, \\ 0 & \text{if } x(\mathrm{I}) = 0 \text{ and } z_\mathrm{J} \leq H\,, \end{cases} \tag{13}$$

so as to "turn off" the (favourable) pressure drop that occurs across the barrier (were that retained, one would need also to retain the delta-function momentum sink). I have dropped the subscript '0' on the eddy viscosity, and the notation "J + 1/2" means that K is to be evaluated at $z_\mathrm{J} + \Delta z/2$. Recall that $\overline{p}_1/\rho_0$ is given by Eq. (9).

We now cast this algorithm into the form

$$c_J\,\overline{u}_1(\mathrm{I},\mathrm{J}+1) +\; b_J\,\overline{u}_1(\mathrm{I},\mathrm{J}) +\; a_J\,\overline{u}_1(\mathrm{I},\mathrm{J}-1) =\; D(\mathrm{I},\mathrm{J}),\quad \mathrm{J} = 1..\mathrm{J_{mx}} \tag{14}$$

where the c_J, b_J, a_J are the "neighbour coefficients", and $\overline{u}_1(\mathrm{I},\mathrm{J})$ is the windspeed perturbation. The coefficients are:

$$a_J \;=\; -\,\frac{\Delta x}{\Delta z}\,K_{\mathrm{J}-1/2}\,, \tag{15}$$

$$b_J \;=\; \Delta z\,\overline{u}_0(\mathrm{J}) \;+\; \frac{\Delta x}{\Delta z}\,K_{\mathrm{J}+1/2} \;+\; \frac{\Delta x}{\Delta z}\,K_{\mathrm{J}-1/2}\,, \tag{16}$$

$$c_J \;=\; -\,\frac{\Delta x}{\Delta z}\,K_{\mathrm{J}+1/2}\,, \tag{17}$$

and

$$D(\mathrm{I},\mathrm{J}) = \Delta z\, \overline{u}_0(\mathrm{J})\, \overline{u}_1(\mathrm{I}-1,\mathrm{J}) \;-\; \mathcal{S}_\mathrm{I}\, \frac{\Delta x\, \Delta z}{\rho_0}\, \frac{\overline{p}_1(\mathrm{I}+1/2,\mathrm{J}) - \overline{p}_1(\mathrm{I}-1/2,\mathrm{J})}{\Delta x}\; . \tag{18}$$

Solve Eqs. (9 and 12-18) numerically to obtain the solution field $\overline{u}_1(x_\mathrm{I}, z_\mathrm{J})$ for a windbreak with height $H = 2$ m standing at $x = 0$ on a surface with roughness length $z_0 = 0.01$ m. Let the upwind friction velocity $u_{*0} = 0.25\,\mathrm{m\,s^{-1}}$ and evaluate $\overline{u}_{0\mathrm{H}}$ accordingly. Use a computational domain bounded by the ground plane ($z = z_0$), an upper plane at $z = z(\mathrm{J_{mx}}) = 20H$, an inflow plane at $x = X_0 = -10H$ and a downwind plane at $x = x(\mathrm{I_{mx}}) = 30H$. Choose your gridlengths as $\Delta x = \Delta z = H/10$.

On a single graph plot your computed transects of $\overline{u}_1$ versus x/H along gridplanes close to these levels: $z/H = (0.4, 0.8, 1.2, 2, 5)$. On the transect at $z/H \approx 0.4$, read off the *minimum* value $\overline{u}_1^{\mathrm{min}}$ and, noting from Eq. (5) that

$$\frac{\overline{u}_1}{\overline{u}_0} \equiv \frac{1}{k_r}\, \frac{\overline{u}(x,z) - \overline{u}_0(z)}{\overline{u}_0(z)}\ , \tag{19}$$

compare your $\overline{u}_1^{\mathrm{min}}/\overline{u}_0$ with the theoretical value given by Wilson et al. (1990), viz.

$$\frac{1}{k_r}\left(\frac{\Delta \overline{u}}{\overline{u}_0}\right)^{\mathrm{min}} = \frac{-1}{(1 + 2\,k_r)^{0.8}} \;\approx\; -1\ . \tag{20}$$

References

Wilson, J.D. 1985. Numerical studies of flow through a windbreak. *J. Wind Eng. Indust. Aero.*, **21**, 119–154.

Wilson, J.D., Swaters, G.E., & Ustina, F. 1990. A perturbation analysis of turbulent flow through a porous barrier. *Q.J.R. Met. Soc.*, **116**, 989–1004.