EAS 471 26 JAN/2016

A. Reconcile RDM with implications of the mass consv. eqn $\frac{\partial c}{\partial t} = -\frac{\partial F_z}{\partial z}$

But $F_z = -K\frac{\partial c}{\partial z}$ with $K = \text{const}$, so $\boxed{\frac{\partial c}{\partial t} = K\frac{\partial^2 c}{\partial z^2}}$

(i) $\int_{-\infty}^{\infty} \frac{\partial c}{\partial t}\, dz = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} c\, dz = K\int_{-\infty}^{\infty}\frac{\partial^2 c}{\partial z^2}\, dz = K\left[\frac{\partial c}{\partial z}\right]_{-\infty}^{\infty} = 0$

Let init cond be that at $t=0$ $\quad c(z,t) = c_0(z) = \delta(z-0)^{*}$ $\quad \left[\frac{1}{m}\right]$

aside

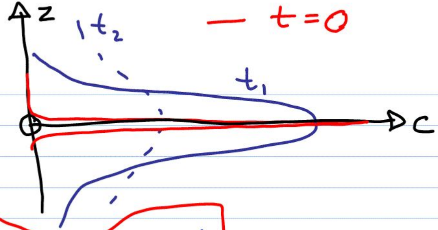
* $\int_{-\infty}^{\infty} \delta(z-z_1)\, dz = \underline{1}$ — I

$\int_{-\infty}^{\infty} \delta(z-z_1)\, f(z)\, dz = f(z_1)$ — II

localizing function
ideally zero width about z_1, and infinite height.

$$\delta(z-z_1) = \lim_{\sigma\to 0} \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(z-z_1)^2/2\sigma^2}}_{\text{Gaussian}}$$

$$\int_{-\infty}^{\infty} c_0(z)\,dz = \int_{-\infty}^{\infty} \delta(z-0)\,dz = 1$$



$[c]$ are $\frac{1}{m}$

But $\frac{\partial}{\partial t}\int_{-\infty}^{\infty} c\,dz = 0$ $\qquad \boxed{\therefore \int_{-\infty}^{\infty} c\,dz = 1 \quad \forall\, t}$

$$z^2 \frac{\partial c}{\partial t} = K z^2 \frac{\partial^2 c}{\partial z^2}$$

$$\int_{-\infty}^{\infty} z^2 \frac{\partial c}{\partial t}\,dz = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} z^2 c\,dz = K\int_{-\infty}^{\infty} z^2 \frac{\partial^2 c}{\partial z^2}\,dz$$

Const

$$= K\left[\cancel{\frac{\partial c}{\partial z}} z^2\right]_{-\infty}^{\infty} \to 0 \;-\; K\int_{-\infty}^{\infty}\left(\frac{\partial c}{\partial z}\, 2z\right)dz = -2K\left[\cancel{c}\, z\right]_{-\infty}^{\infty} \to 0 \;+\; 2K\int_{-\infty}^{\infty} c\,dz$$

$$\int_{-\infty}^{\infty} z \frac{\partial c}{\partial t}\,dz = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} zc\,dz = \frac{\partial \bar{z}}{\partial t} = K\int_{-\infty}^{\infty} z \frac{\partial^2 c}{\partial z^2} = \cancel{K\left[\frac{\partial c}{\partial z} z\right]_{-\infty}^{\infty}} \to 0$$

$$- K\int \frac{\partial c}{\partial z}\,dz$$

$$\frac{\partial \bar{z}}{\partial t} = -K\left[c\right]_{-\infty}^{\infty} = 0 \quad \text{and} \quad \bar{z}_0 = 0 \quad \left[\int_{-\infty}^{\infty} z \underbrace{\delta(z-0)}_{c_0(z)}\, dz = 0\right]$$ ③

$$\sigma_z^2 \equiv \int_{-\infty}^{\infty} z^2 c(z)\, dz - \left(\int_{-\infty}^{\infty} z\, c(z)\, dz\right)^2$$

mean square — square of mean.

But we showed that $\frac{\partial}{\partial t}\int_{-\infty}^{\infty} z^2 c(z)\, dz = 2K$

$$\therefore \frac{\partial \sigma_z^2}{\partial t} = 2K$$

$$\sigma_z^2 = 2Kt + a$$

But at $t = 0$ $\sigma_z^2 = 0$ $\therefore a = 0$

$$\boxed{\sigma_z = \sqrt{2Kt}}$$

Characteristic of diffusion "parabolic spread"

B Reynolds-averaged consv. eq^n $\quad \dfrac{\partial \phi}{\partial t} = -\nabla \cdot \vec{F}_\phi + Q$

$$\frac{\partial \bar{\phi}}{\partial t} = -\frac{\partial \overline{F}_{\phi x}}{\partial x} - \frac{\partial \overline{F}_{\phi y}}{\partial y} - \frac{\partial \overline{F}_{\phi z}}{\partial z} + \overline{Q}$$

$$\overline{F}_{\phi x} = \overline{u\phi} - K\frac{\partial \bar{\phi}}{\partial x}$$ (no radiative transport) (Let $Q = 0$ too)

$$= \underbrace{\bar{u}\,\bar{\phi}}_{\text{convective transp. by the mean flow}} + \underbrace{\overline{u'\phi'}}_{\text{conv. transp by the unresolved (or turbulent) flow}} - \underbrace{K\frac{\partial \bar{\phi}}{\partial x}}_{\text{diffusion}}$$

neglect

$$\frac{\partial \bar{u}\bar{\phi}}{\partial x} = \bar{u}\frac{\partial \bar{\phi}}{\partial x} + \bar{\phi}\frac{\partial \bar{u}}{\partial x} \quad \text{etc} \ldots$$

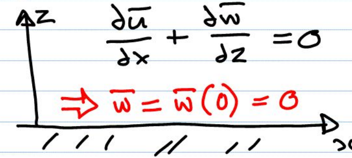
$$\frac{\partial \bar{\phi}}{\partial t} + \bar{u}\frac{\partial \bar{\phi}}{\partial x} + \bar{v}\frac{\partial \bar{\phi}}{\partial y} + \bar{w}\frac{\partial \bar{\phi}}{\partial z} = \overbrace{-\frac{\partial \overline{u'\phi'}}{\partial x} - \frac{\partial \overline{v'\phi'}}{\partial y} - \frac{\partial \overline{w'\phi'}}{\partial z}}^{\text{eddy flux divergence}}$$

$$- \bar{\phi}\underbrace{\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z}\right)}_{\text{velocity divergence, assume zero}}$$

Even if $\bar{u}, \bar{v}, \bar{w}$ known as function of x, y, z, t still have "closure problem"

Adopt eddy diffusion closure

Let's narrow the problem to a horiz. homog.* and stationary* ASL with coords. chosen such that $\bar{v} = 0$



$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} = 0$$

$$\Rightarrow \bar{w} = \bar{w}(0) = 0$$

Fact $\sigma_u \ll \bar{u}$ provided $z \gg z_0$

so let's neglect $\overline{u'\phi'}$ relative to $\bar{u}\,\bar{\phi}$

Assume

$$\left.\begin{aligned} \overline{w'\phi'} &= -K_{ez}\frac{\partial \bar{\phi}}{\partial z} \\ \overline{v'\phi'} &= -K_{ey}\frac{\partial \bar{\phi}}{\partial y} \end{aligned}\right\}$$

eddy diffusivites $[m^2 s^{-1}]$

will drop the "e", $K_z(z)$, $K_y(z)$

missing minus signs

$$\frac{\partial \bar{\phi}}{\partial t} + \bar{u}(z)\frac{\partial \bar{\phi}}{\partial x} = -\frac{\partial}{\partial y}\left[K_y(z)\frac{\partial \bar{\phi}}{\partial y}\right] - \frac{\partial}{\partial z}\left[K_z(z)\frac{\partial \bar{\phi}}{\partial z}\right]$$

crosswind dispersion   vertical dispersion

* stats. of velocity indep of x, y, t

Let XWIC be $\bar{c} = \int_{-\infty}^{\infty} \bar{\Phi}\, dy$, and assume a steady source

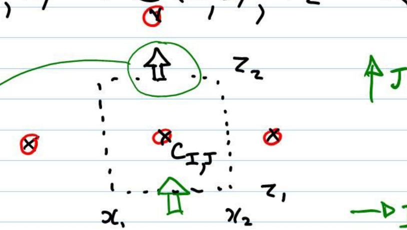
Then $\bar{u}\,\dfrac{\partial \bar{c}}{\partial x} = \dfrac{\partial}{\partial z}\left[K_z \dfrac{\partial \bar{c}}{\partial z} \right]$

governs XWIC from a point source
or/and crosswind conc due to a line source

C Discretizing this eq^n using a control volume method

To simplify terminology let $\bar{u}(z) \to U(z)$, $\bar{c}(x,z) \to C(x,z)$, $K_z \to K$

$$\frac{\partial}{\partial x} UC = \frac{\partial}{\partial z}\left[K \frac{\partial C}{\partial z} \right]$$



Integrate throught the control volume

$$\int_{z_1}^{z_2} \left[UC \right]_{x_1}^{x_2} dz = \int_{x_1}^{x_2} \left[K \frac{\partial C}{\partial z} \right]_{z_1}^{z_2} dx$$

eddy diffusivity at base of control volume

$$\Delta z\, U_J \left[C_J \right]_{x_1}^{x_2} = \Delta x \left[K_{J+1/2} \frac{C_{I,J+1} - C_{I,J}}{\Delta z} - K_{J-1/2} \frac{C_{I,J} - C_{I,J-1}}{\Delta z} \right]$$

Our x axis is "1-way" (no $\frac{\partial^2}{\partial x^2}$ term), stuff only goes downwind. Therefore we can represent

$$\left[C_J \right]_{x_1}^{x_2} \quad \text{as} \quad C_{I,J} - C_{I-1,J}$$

$$\Delta z \, U_J \left[C_{I,J} - C_{I-1,J} \right] = \Delta x \, K_{J+1/2} \frac{C_{I,J+1} - C_{I,J}}{\Delta z}$$

$$- \Delta x \, K_{J-1/2} \frac{C_{I,J} - C_{I,J-1}}{\Delta z}$$

(Have chosen a uniform grid, $\Delta x = \text{const}$
$\Delta z = \text{const}$)