EAS 471 Continuing with Discretized Adv.-Diffusion type eqns 28 JAN 2016

$$\Delta z\, U_J \left[C_{I,J} - C_{I-1,J} \right] = \Delta x \left[K_{J+1/2} \frac{C_{I,J+1} - C_{I,J}}{\Delta z} - K_{J-1/2} \frac{C_{I,J} - C_{I,J-1}}{\Delta z} \right]$$

Collect terms. Anything "known" goes on rhs

$$C_{I,J+1} \underbrace{\left[\frac{\Delta x}{\Delta z} K_{J+1/2} \right]}_{\text{"}\mathbb{C}\text{" coefft}} + C_{I,J} \underbrace{\left[-\Delta z\, U_J - \frac{\Delta x}{\Delta z} K_{J+1/2} - \frac{\Delta x}{\Delta z} K_{J-1/2} \right]}_{\text{"}\mathbb{B}\text{" coefft}}$$

$$+ C_{I,J-1} \underbrace{\left[\frac{\Delta x}{\Delta z} K_{J-1/2} \right]}_{\text{"}\mathbb{A}\text{" coefft}} = \underbrace{-\Delta z\, U_J\, C_{I-1,J}}_{\text{"}\mathbb{D}\text{"}}$$

column matrix

column matrix of RHS

$$\mathbb{M}\, \mathbb{C} = \mathbb{D} \quad \therefore \mathbb{C} = \mathbb{M}^{-1} \mathbb{D}$$

square matrix of coeffts

Simpler example

$$\frac{C_I^{n+1} - C_I^n}{\Delta t} = \mathcal{D} \frac{C_{I+1}^{n+1} + C_{I-1}^{n+1} - 2 C_I^{n+1}}{\Delta x^2} \quad \left(\frac{\partial c}{\partial t} = \mathcal{D} \frac{\partial^2 c}{\partial x^2} \right)$$

$$C_{I+1}^{n+1}\left[\frac{D\Delta t}{\Delta x^2}\right] + C_{I}^{n+1}\left[-1-2\frac{D\Delta t}{\Delta x^2}\right] + C_{I-1}^{n+1}\left[\frac{D\Delta t}{\Delta x^2}\right] = C_I^n$$

Let $\frac{D\Delta t}{\Delta x^2} = \frac{1}{4} \quad \left(< \frac{1}{2}\right)$, assume a 3 point grid

B/conds: $C_1^n = \frac{1}{2}$, $C_3^n = -1$

Init cond: $C_1^1 = \frac{1}{2}$, $C_3^1 = -1$, $C_2^1 = 1$

$x=$	-1	0	1
$I=$	1	2	3

At $t \to \infty$ expect linear solution, so $C_2^\infty = (1/2 - 1)/2 = -1/4$

$$C_{\underset{\downarrow\, 3}{I+1}}^{n+1 \to 2}\left[\frac{1}{4}\right] + C_{\underset{\downarrow\, 2}{I}}^{n+1 \to 2}\left[-\frac{3}{2}\right] + C_{\underset{\downarrow\, 1}{I-1}}^{n+1 \to 2}\left[\frac{1}{4}\right] = C_{\underset{\downarrow\, 1}{I}}^{n \to 1}$$

for $I = 2$

Notice the coefft. matrix is tridiagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/4 & -3/2 & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ -1 \end{bmatrix}$$

$MC = D$

Matlab solution

$C = M \backslash D$

replace for next time step

By hand: * row 2 by 4

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -6 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 4 \\ -1 \end{pmatrix}$$

Subtract row 1 from row 2

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -6 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} \begin{pmatrix} 1/2 \\ 3\,1/2 \\ -1 \end{pmatrix}$$

Subtract row 3 from row 2

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} \begin{pmatrix} 1/2 \\ 4\,1/2 \\ -1 \end{pmatrix}$$

$$\frac{9}{2} \cdot \frac{-1}{6} = \frac{-9}{12}$$

Divide row 2 by -6

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -3/4 \\ -1 \end{bmatrix}$$

$C_2^2 = -3/4$