EAS 471

3 MAR 2016

G.I. Taylor's Lagrangian theory for dispersion in unbounded homog. turbulence

Suppose the Eulerian veloc. along z has zero mean, and st. dev σ_w

Consider statistics of particles released independently at $z = t = 0$.

$$Z(t) = \int_0^t W(t')\,dt' \quad \text{where} \quad W(t') = \left(dZ/dt\right)_{t=t'}$$

$\langle Z(t) \rangle$ = ensemble mean Z at time t after release.

$$= \left\langle \int_0^t W(t')\,dt' \right\rangle = \int_0^t \langle W(t') \rangle \, dt' = 0$$

$$\frac{dZ^2}{dt} = 2Z\frac{dZ}{dt} = 2\,W(t)\int_0^t W(t')\,dt'$$

✱ see p7

$$= 2\int_0^t W(t)\,W(t')\,dt'$$

$$\left\langle \frac{dZ^2}{dt} \right\rangle = \frac{d}{dt}\langle Z^2 \rangle = 2\int_0^t \langle W(t)\,W(t') \rangle \, dt' \qquad \text{— I}$$

Since $\langle Z \rangle = 0 \; \forall t$, $\langle Z^2 \rangle = \sigma_z^2$ (variance of displacement)

Define $p(z, t \,|\, 0, 0)\, dz$ = probability that a particle released at $z = 0$ at $t = 0$ lies in $z \pm dz/2$ at time t

Clearly $\int_{-\infty}^{\infty} p \, dz = 1$ for all t, so p is a prob. density func.

Furthermore $p(z, 0 \,|\, 0, 0) = \delta(z - 0) \quad \left[\frac{1}{m}\right]$

Loosely, can think of p as "concentration"

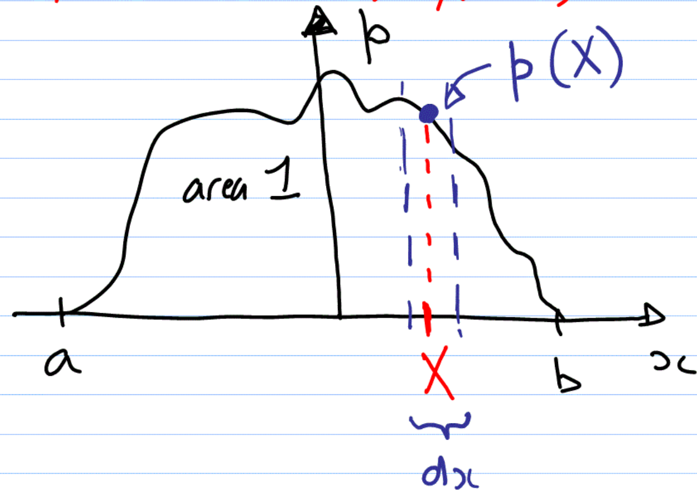
$$\langle Z \rangle = \int_{-\infty}^{\infty} z \, p(z, t \,|\, 0, 0) \, dz$$

An aside on probability density functions (cts random variables)

Let x be a CTS random variable defined on range $a \le x \le b$

Then the probab. that a single sample of x is i.e. X is

$$P(x \le X) = \int_a^X p(x)\,dx$$



or equally,

$$P\left(X - \frac{dx}{2} \le x \le X + \frac{dx}{2}\right)$$

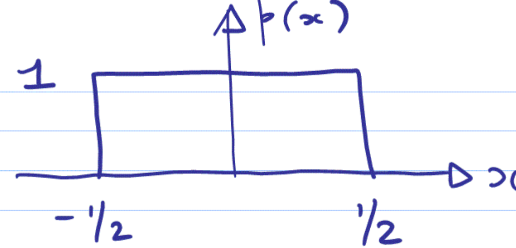
$$= p(X)\,dx$$

$$\overline{x} = \int_a^b x\,p(x)\,dx\,, \quad \overline{x^2} = \int_a^b x^2 p(x)\,dx \quad \text{etc...}$$

For any function $g(x)$, the mean value of g is

$$\overline{g} = \int_a^b g(x)\,p(x)\,dx$$

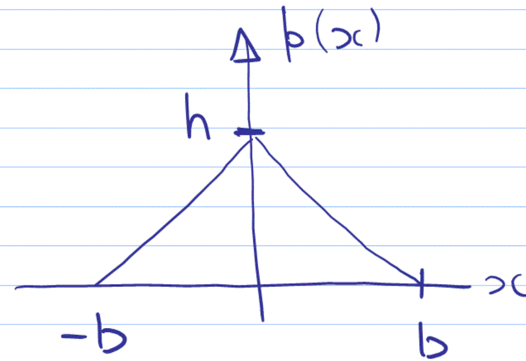
Examples:



$$\overline{p} = ? = \int_{-1/2}^{1/2} x \, dx = \left[\frac{x^2}{2}\right]_{-1/2}^{1/2} = \frac{1}{8} - \frac{1}{8} = 0$$

$$\overline{p^2} = ? = \int_{-1/2}^{1/2} x^2 \, dx = \left[\frac{x^3}{3}\right]_{-1/2}^{1/2}$$

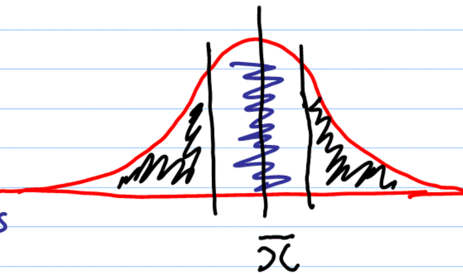
$$= \frac{1}{24} - \left(-1/24\right) = 1/12$$



Compute variance if $h = 2$ $\left(\begin{matrix} b = 1/2. \\ \text{why?} \end{matrix}\right)$

Normal (Gaussian) distrib,

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left[\frac{-(x-\bar{x})^2}{2\sigma_x^2}\right].$$



Three equal areas define three equi-probable classes

If X is the median of x, then $\int_a^X p(x)\,dx = \int_X^b p(x)\,dx$

The mode Y is the value of x for which $p(x)$ is maximal.

Back to eq I

Define $\xi = t - t'$ such that $d\xi = -dt'$

$$\frac{d}{dt}\langle Z^2\rangle = 2\int_t^0 \langle W(t'+\xi)\,W(t')\rangle\,(-d\xi)$$

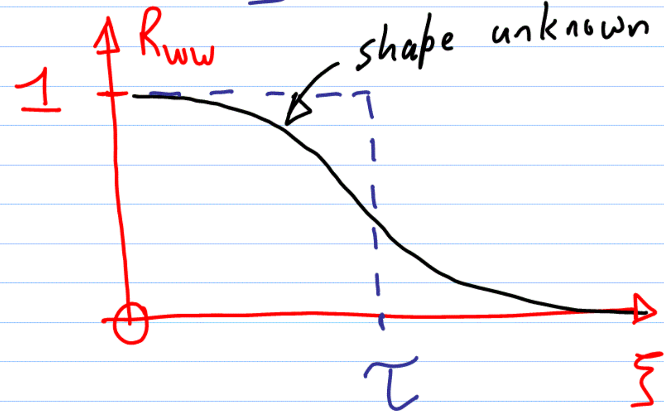
$$= 2\int_0^t \langle W(t')\,W(t'+\xi)\rangle\,d\xi$$

$$= 2\sigma_W^2 \int_0^t R_{WW}(\xi, t')\,d\xi \quad \text{where}$$

time lag

$$R_{WW}(\xi, t') = \langle W(t')\,W(t'+\xi)\rangle / \sigma_W^2$$

R_{ww} is a statistic, and we'll assume it is stationary so not a function of t'



τ = Lagrangian integral timescale

$$= \int_0^\infty R_{ww}(\xi)\, d\xi$$

= a measure of velocity persistence

$$\frac{d}{dt}\langle Z^2\rangle = \frac{d\sigma_z^2}{dt} = 2\sigma_w^2 \int_0^t R(\xi)\, d\xi \xrightarrow{t \gg \tau} 2\sigma_w^2 \tau$$

i.e. for $t \gg \tau$ $\quad \dfrac{d\sigma_z^2}{dt} = 2K_\infty$, $\quad K_\infty = \sigma_w^2 \tau$

far field eddy diffusivity

We saw earlier that for the RDM and for the diffusion eqn that

$$\frac{d\sigma_z^2}{dt} = 2K \qquad \left(\sigma_z^2 = 2Kt \right)$$

The far field diffusivity has been related to Lagr. velocity statistics. (It's a property of the motion).

‡ Footnote from page 1: <u>why one can exchange order of time differentiation and ensemble averaging</u>. Suppose the ensemble average is an average over N paths, labelled $i = 1 \ldots N$. Then

$$\left\langle \frac{dG}{dt} \right\rangle = \frac{1}{N}\left[\frac{dG_1}{dt} + \frac{dG_2}{dt} + \ldots + \frac{dG_N}{dt} \right]$$

$$= \frac{1}{N}\frac{d}{dt}\left[G_1 + G_2 + \ldots + G_N \right]$$

$$= \frac{d}{dt}\frac{1}{N}\left[G_1 + G_2 + \ldots + G_N \right] = \frac{d}{dt}\langle G \rangle$$

for any property G defined on the trajectories, G_1 on the first, G_2 on the 2nd