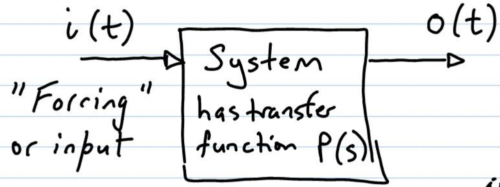
EAS 471 <u>The "waves" viewpoint in math/phys/eng.</u> 4 FEB 2016

(A)



special inputs

(i) step response $i(t) = h(t-0)$

(ii) impulse response $i(t) = \delta(t-0)$

(iii) sinusoidal response $i(t) = A_i \sin 2\pi f t$

For a "linear system*" if you know $o(t)$ for any one of those special inputs, you know response to an <u>arbitrary</u> input

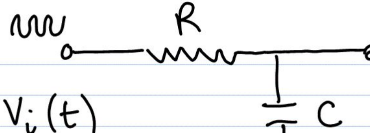
* governing eq^n is linear

$$\text{Transfer function } P(s) = \frac{\text{Laplace transf. of } o(t)}{\text{LT of } i(t)}$$

where for any function $f(t)$ the LT is

$$L[f(t)] = \bar{f}(s) = \int_0^\infty f(t)\, e^{-st}\, dt\,,$$ s the "complex freq."

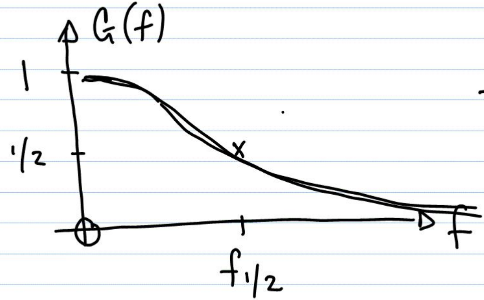
Example‡



$V_i(t) = A_i \sin 2\pi f t$ $\qquad$ $V_o(t) = A_o \sin(2\pi f t + \phi)$

‡ lowpass RC filter (analogous to a thermometer with thermal inertia to which heat flows by imperfect conduction)

Power gain $G(f) = \dfrac{A_o^2}{A_i^2} = \left|P(j\omega)\right|^2 = \dfrac{1}{1 + (f/f_{1/2})^2}$



$$f_{1/2} = \frac{1}{2\pi RC}$$

(B) Suppose function $\phi = \phi(x)$ is governed by $L\phi = 0$

where $L = a + b\dfrac{\partial}{\partial x} + c\dfrac{\partial^2}{\partial x^2} + \ldots$ (linear operator)

Then if $\phi_1(x)$, $\phi_2(x)$ are solutions, so is $\phi_1 + \phi_2$. Whence the idea of representing an arbitrary solution as a sum or "superposition" of elementary "basis functions" (or "waves"). Very typically eq^{n}s like $L\phi = 0$ ("homogeneous linear d.e.) have solutions like $e^{jmx/L}$ $\left(= \cos m\frac{x}{L} + j \sin m \frac{x}{L} \right)$

(C) Fourier series: let $f(x)$ be defined on $-L \le x \le L$, then it can be represented as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

where

$$a_n = \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, d\frac{x}{L}$$

b_n sin

(D) Fourier Transform Pair.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{jkx}\, dk$$

summation of ∞ set of (complex) waves

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-jkx}\, dx$$

$$\begin{cases} k = \dfrac{2\pi}{\lambda} & \text{waveno.} \\ j = \sqrt{-1} \end{cases}$$

(E) Example of wave-wave interaction in a non-linear system

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \quad \text{and let } u = A\sin(\omega t + kx)$$

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} \quad \text{where } \frac{\partial u}{\partial x} = Ak\cos(\omega t + kx)$$

$$= -A^2 k \underbrace{\sin(\omega t + kx)}_{A} \underbrace{\cos(\omega t + kx)}_{B}$$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$ (5)

$\sin(A-B)$ $\qquad -$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\frac{\partial u}{\partial t} = \frac{-A^2 k}{2}\left[\sin\left(2\omega t + 2kx\right) + \sin 0\right]$$

Thus
$$u(t+\Delta t) \approx u(t) - \frac{A^2 k \Delta t}{2} \sin 2(\omega t + kx)$$

sol^n has components at ω, k and $2(\omega, k)$

At $t + 2\Delta t$ it will contain $4\times, 3\times, 2\times, 1\times$

At $t + 3\Delta t$ $\qquad 8, 7, 6, 5, 4, 3, 2, 1$

numeric stability (versus Courant number) of this algorithm:

Test
$$\frac{\phi_I^{n+1} - \phi_I^n}{\Delta t} = -u \frac{\phi_{I+1}^n - \phi_{I-1}^n}{2\Delta x}$$

$\phi(x, 0) = \sin \pi x$

on $-1 \le x \le 1$ with $\phi(1, t) \equiv \phi(-1, t)$ and