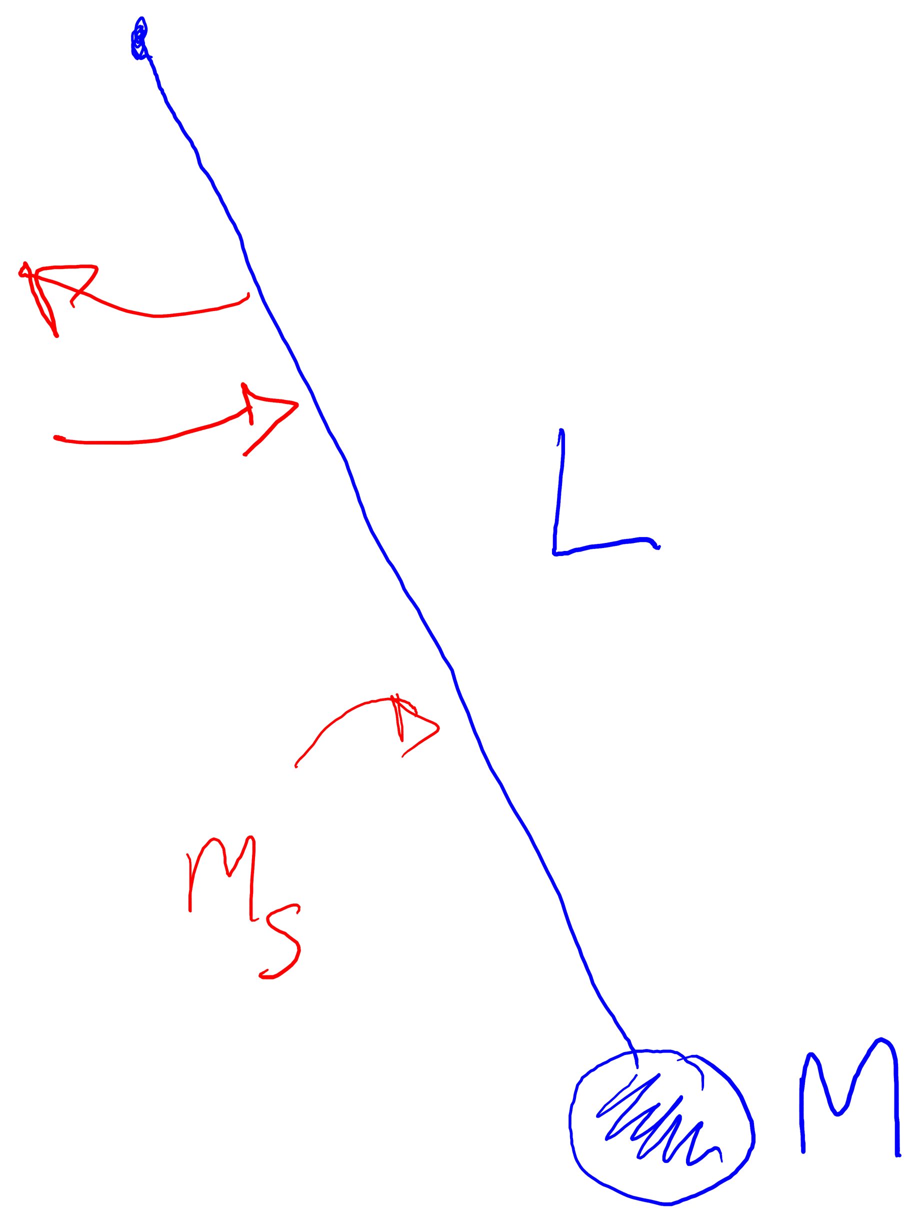
EAS 471 Dimensional Analysis (ctd) 7 Jan 2016

Pendulum Suppose we want to know g the period T of the pendulum.



fluid
viscosity* ν
density ρ

$\boxed{A}$ Guess T is controlled by M, g, L

$n = 4 \qquad m = 3 \qquad (kg, m, s)$

$n - m = 1 \qquad R = \text{const.}$ (R dimensionless)

The only scale avail for time is (by inspection) $\left[\frac{g}{L}\right]^{-1/2}$

$$\frac{T}{\sqrt{L/g}} = \text{const.}$$

Can automate finding ratios as follows ("method of indices")

* kinematic viscosity $[m^2 s^{-1}]$

Example $\boxed{B}$ (I'll add M_s, ν so $n=6$ $m=3$) (can guess m/M_s will arise)

no units

Assume $T = \sum_{i=1}^{\infty} \left[C_i\, L^{\alpha_i} M^{\beta_i} g^{\gamma_i} M_s^{\delta_i} \nu^{\epsilon_i} \right]$

$$m: \quad 0 = \alpha + \gamma + 2\epsilon \qquad \text{I}$$

$$kg: \quad 0 = \beta + \delta \qquad \text{II}$$

$$s: \quad 1 = -2\gamma - \epsilon$$

3 eqns in 5 unknowns

$$\gamma = \tfrac{1}{2}(-1-\epsilon) \qquad \text{III}$$

Elim. γ from I using III

$$0 = \alpha - \tfrac{1}{2}(1+\epsilon) + 2\epsilon = \alpha - \tfrac{1}{2} + \tfrac{3}{2}\epsilon$$

$$\alpha = \tfrac{1}{2} - \tfrac{3}{2}\epsilon$$

ϵ, β

$$\gamma = -\tfrac{1}{2} - \tfrac{1}{2}\epsilon \qquad \delta = -\beta$$

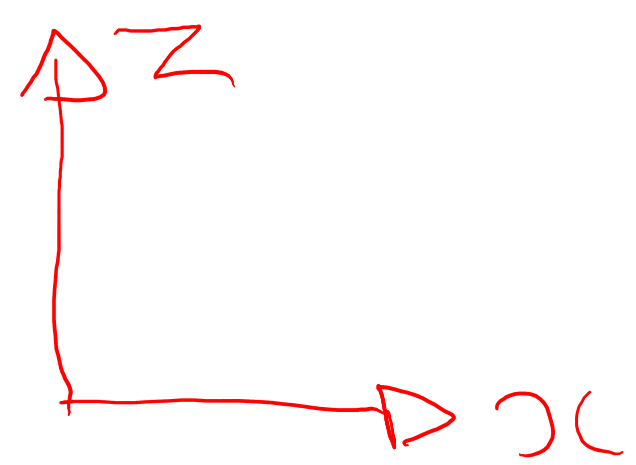
$$T = \sum c_i \, L^{\frac{1}{2} - \frac{3}{2}\epsilon_i} \, M^{\beta_i} \, g^{-1/2 - \frac{1}{2}\epsilon_i} \, m_s^{-\beta_i} \, \nu^{\epsilon_i}$$

$$= \sum c_i \sqrt{L/g} \left(\frac{\nu}{L^{3/2} g^{1/2}} \right)^{\epsilon_i} \left(\frac{m}{m_s} \right)^{\beta_i}$$

$$\frac{T}{\sqrt{L/g}} = f\left(\frac{m}{m_s} , \frac{\nu}{L^{3/2} g^{1/2}} \right) \xrightarrow{m_s \to 0,\, \nu \to 0} 2\pi$$

Newton's analysis with friction neglected and $m_s \to 0$ and small angle (small amplitude): $T = 2\pi \sqrt{L/g}$

Back to our idealized dispersion problem



Idealizations: 2D

Horiz. homog, so stats. indep of x

Stationary flow " of wind " of t

Let $\underset{\sim}{u} = (\bar{u}, w)$

w is turbulent with $\bar{w} = 0$

$\bar{u}$ fixed in time but might vary with z

Let's "mimic" transport from this source to elsewhere z

can we compute trajectories?

Particle position: $X = X_0 + \sum \Delta X$, $\Delta X = \bar{u} \, \Delta t$

$$Z = Z_0 + \sum \Delta Z$$

Random displacement model, or random walk in position

We'll let Δt = const. In unbounded homogeneous turbulence

$$\Delta Z = \sqrt{2K\Delta t}\; r \qquad r \in \underset{\text{Normal}}{N}(0, 1)$$

(Gaussian)

mean, variance

K: eddy diffusivity $[m^2 s^{-1}]$

G.I. Taylor (1921) proved that

$$K = \sigma_w^2 \, \tau$$

σ_w^2: variance of vert. veloc.

τ: Lagrangian integral time scale of the turbulence

<u>Solution</u>

$$Z(n\Delta t) = \sum_1^n \Delta Z + Z_0 .$$

$$= Z_0 + \sqrt{2K\Delta t}\,\left(r_1 + r_2 + r_3 + \ldots r_n\right)$$

Let this be repeat for M paths (M particles)

$$\overline{Z}(n\Delta t) = Z_0 + \overbrace{\sqrt{2K\Delta t}}^{\text{"B"}}\left(\overline{r_1} + \overline{r_2} + \overline{r_3} + \ldots + \overline{r_n}\right)$$

$$\xrightarrow[\infty]{M} Z_0$$

(overbar represents mean over M independent paths)

$$Z^2(n\Delta t) = \left[Z_0 + B\left(r_1 + r_2 + \ldots + r_n\right)\right]^2$$

$$= Z_0^2 + B^2\left(r_1^2 + r_2^2 + \ldots + r_n^2\right) + 2B^2\left(r_1 r_2 + r_1 r_3 + \ldots\right)$$

$$+ 2BZ_0\left(r_1 + r_2 + \ldots + r_n\right)$$

Now average, remembering r_1 is indep of $r_2, r_3, \ldots$

$$\overline{Z^2}(n\Delta t) = Z_0^2 + B^2\left(\overline{r_1^2} + \overline{r_2^2} + \dots + \overline{r_n^2}\right)$$

$$+ 2B^2\left(\overline{r_1 r_2} + \overline{r_1 r_3} + \dots\right)$$

$$+ 2BZ_0\left(\overline{r_1} + \overline{r_2} + \dots + \overline{r_n}\right)$$

covariance (zero)

$$\xrightarrow[\mathrm{d}]{\mathrm{M}} Z_0^2 + nB^2 + 0 + 0$$

$$= Z_0^2 + 2nK\Delta t$$

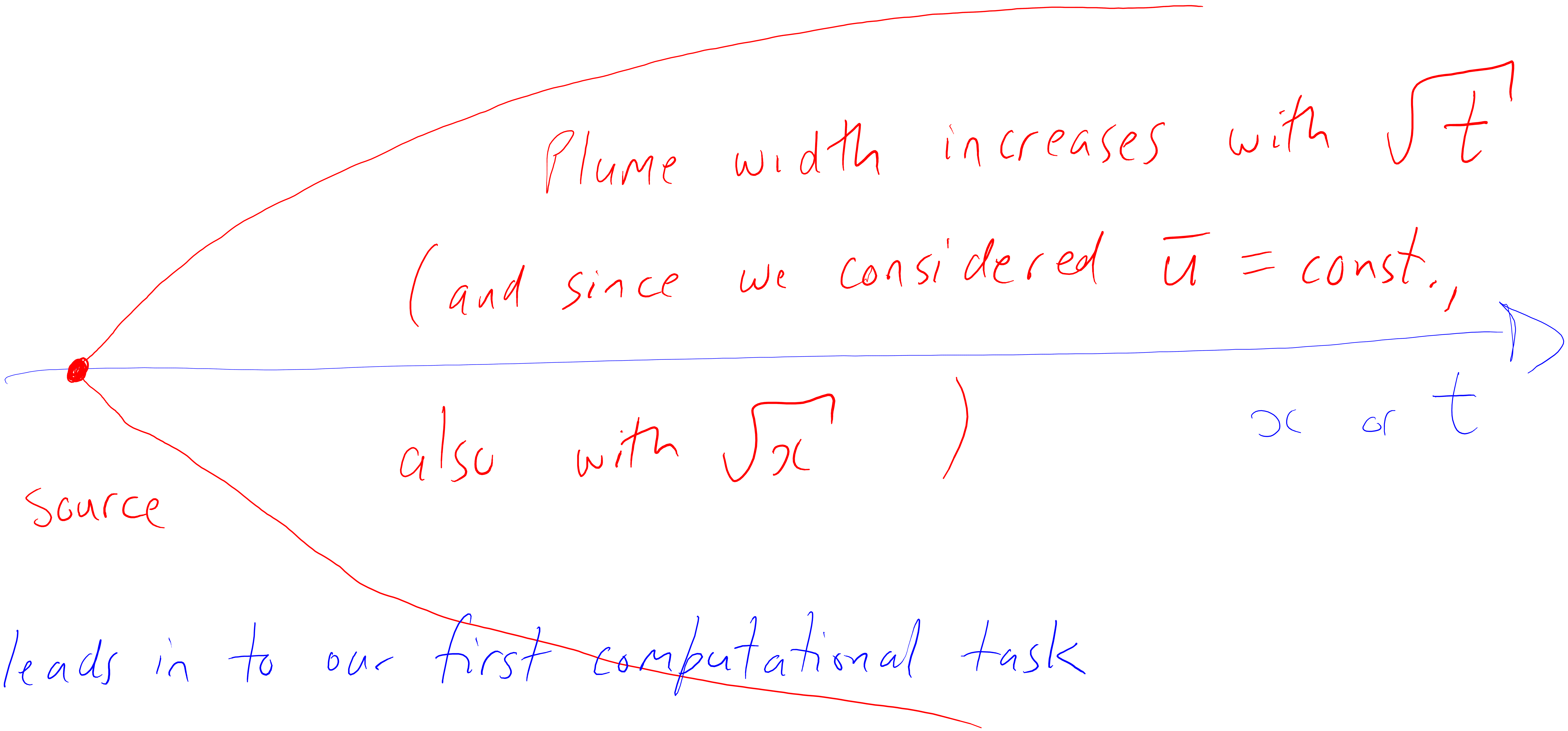
$$= Z_0^2 + 2Kt$$

Let $Z_0 = 0$ (or, if $Z_0 \neq 0$, carry out same exercise to find $\overline{(Z-Z_0)^2} = 2Kt$)

Mean square displacement $\overline{z^2}(t) = 2Kt$

Root mean square displacement $\sigma_z = \sqrt{2Kt}$

"Parabolic spread in time" z characteristic of "diffusion"



Plume width increases with $\sqrt{t}$

(and since we considered $\overline{u} = \text{const.}$, also with $\sqrt{x}$)

This leads in to our first computational task