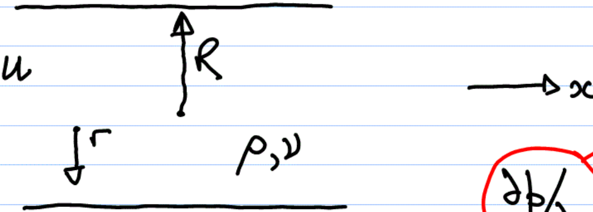
(A) Exercise in dimensional analysis.

Law for $u = u(r)$ in laminar, incompressible* steady state, unidirectional pipe flow (fluid density ρ and kinematic viscosity ν, pressure gradient $\partial p/\partial x$)



$\partial p/\partial x$ constant forcing, (negative)

$n = 6, \quad m = 3$

Classify variables:

variable of interest	u
coordinate	r, x [drop x]
fluid	ρ, ν
forcing	$\partial p/\partial x$
geometry of domain	R

$$* \quad \nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \underbrace{\cancel{\frac{\partial v}{\partial y}}}_{0} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0} = 0$$

u is indep of x

$$\frac{uR}{\nu} = f(\Pi_1, \Pi_2) = f\left(\frac{r}{R}, \Pi_2\right). \qquad \Pi_2 = \frac{R}{\rho v^2}\frac{\partial p}{\partial x},$$

but what is v? $\quad v = \nu/R \qquad \nu/R$ is [$m\ s^{-1}$]

Thus $\Pi_2 = \frac{R^3}{\rho \nu^2} \frac{\partial p}{\partial x}$

Check Eqn of motion for u-mtm is

$$\frac{Du}{Dt} = 0 = \cancel{\frac{\partial u}{\partial t}}^{\,0} + u \cancel{\frac{\partial u}{\partial x}}^{\,0}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u$$

Laplacian in cylindrical coord. is $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right)$

$$\frac{\partial}{\partial r}\left[r \frac{\partial u}{\partial r}\right] = \frac{1}{\rho \nu}\frac{\partial p}{\partial x} r = F r \qquad (F = \text{const.})$$

$$r\frac{\partial u}{\partial r} = \frac{1}{2} F r^2 + c_1 \implies \frac{\partial u}{\partial r} = \frac{1}{2} F r + c_1/r$$

$$u = \frac{1}{4} F r^2 + c_1 \ln r + c_2 \quad \begin{cases} u = 0 & \text{on } r = R \quad (\text{no slip}) \\ u = U_0 & \text{on } r = 0 \end{cases}$$

Conclude $c_1 = 0$, $\quad u = \frac{1}{4} F (r^2 - R^2) \qquad \frac{uR}{\nu} = \frac{1}{4}\frac{1}{\rho\nu}\frac{\partial p}{\partial x}(r^2 - R^2)\frac{R}{\nu}$

$$\frac{uR}{\nu} = -\frac{1}{4}\frac{R^3}{\rho\nu^2}\frac{\partial p}{\partial x}\left(1-\left(\frac{r}{R}\right)^2\right) = f\left(\frac{r}{R}, \frac{R^3}{\rho\nu^2}\frac{\partial p}{\partial x}\right)$$

as anticipated.

B Continue G.I. Taylor's theory

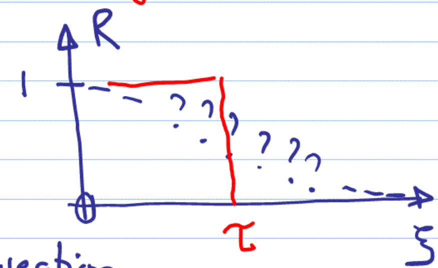
$$\frac{d\langle Z^2\rangle}{dt} = 2\sigma_w^2\int_0^t R(\xi)\,d\xi = 2K, \quad \therefore K = \sigma_w^2\int_0^t R(\xi)\,d\xi$$

Implications ① K not a property of the flow alone

② $K \xrightarrow[\ll \tau]{t} 0$

③ "Diffusion solution" $\sigma_z = \sqrt{2Kt}$ wrong for small t

$$\boxed{\langle Z^2\rangle = \sigma_z^2 = 2\sigma_w^2\int_0^t (t-\xi)\,R(\xi)\,d\xi}$$



④ Eddy diffusion is not a fundamental paradigm for turbulent convection

If $R(\xi) = e^{-\xi/\tau}$ (a common approx) then

$$\frac{\sigma_z^2}{2\sigma_w^2} = t\tau - \tau^2\left(1 - e^{-t/\tau}\right)$$

Asymptotic limits:

① Near field, $t \ll \tau$ $\qquad K = \sigma_w^2 \int_0^t R(\xi)\,d\xi \simeq \sigma_w^2 \int_0^t 1\,dt = \sigma_w^2 t$

$$\frac{d\sigma_z^2}{dt} = 2K = 2\sigma_w^2 t$$

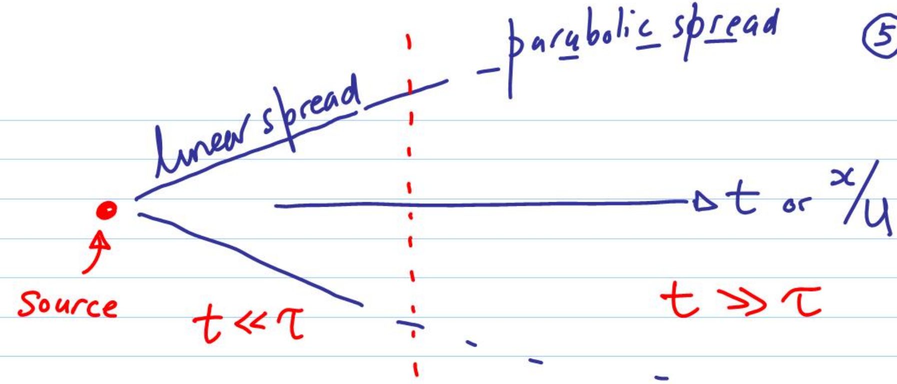
$\sigma_z^2 = \sigma_w^2 t^2$, $\sigma_z = \sigma_w t$ linear spread in near field

"memory dominated" region

② Far field $t \gg \tau$ $\qquad \dfrac{d\sigma_z^2}{dt} \simeq 2\sigma_w^2 \tau = 2K_\infty$

$\sigma_z^2 = 2K_\infty t$, K_∞ the "far field diffusivity"

This was restricted to unbounded, homog, stationary turbulence.



C MODERN GENERALIZATION

The fundamental advance was D.J. Thomson (1987, Quart. J. R. Met. Soc.)

Basic form of "generalized Langevin model"

random forcing

$$dU_i = \underbrace{a_i(\vec{X}, \vec{U}, t)\,dt}_{\text{deterministic term}} + \underbrace{b_{ij}(\vec{U}, \vec{X}, t)}_{\text{random term}}\, d\xi_j$$

Einstein summation

$b_{i1}\, d\xi_1 + b_{i2}\, d\xi_2 + b_{i3}\, d\xi_3$

where $d\xi_j$ are Gaussian with variance dt

Aside: Langevin's model of Brownian motion (in 1D)

Gaussian   mean   variance

$$dW = -\frac{W}{\tau}\,dt + c\,r \quad , \quad r \in N(0, 1)$$