EAS 471 — Applications of the "waves" representation — 9 FEB 2016

## A) Action of the Laplacian* as lowpass filter

$$\frac{\partial T}{\partial t} = \underbrace{K}_{\text{thermal diffusivity}} \frac{\partial^2 T}{\partial x^2}$$

and invoke "waves representation" $T(x,t) = \int_{-\infty}^{\infty} \hat{T}(k,t)\, e^{j\overset{\text{wavenumber}}{k} x}\, dk$

$$\frac{\partial T}{\partial t} = \int_{-\infty}^{\infty} \frac{\partial \hat{T}}{\partial t} e^{jkx}\, dk$$

$$\frac{\partial T}{\partial x} = \int_{-\infty}^{\infty} \hat{T}(k,t)\,(jk)\, e^{jkx}\, dk$$

$$\int_{-\infty}^{\infty} \left( \frac{\partial \hat{T}}{\partial t} - K (jk)^2 \hat{T} \right) e^{jkx}\, dk = 0$$

implies $\dfrac{\partial \hat{T}}{\partial t} + k^2 K \hat{T} = 0$

Define $T(x,0) = \int_{-\infty}^{\infty} \hat{T}_0(k)\, e^{jkx}\, dk$

$$\frac{1}{\hat{T}} \frac{\partial \hat{T}}{\partial t} = \frac{\partial \ln \hat{T}}{\partial t} = -k^2 K$$

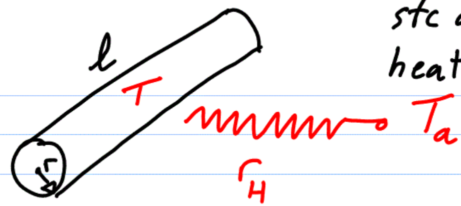
$$\ln \hat{T} = -\underbrace{k^2 K t}_{(m^{-1})^2\ m^2 s^{-1}\ s} + c$$

$$\hat{T} = e^{-k^2 K t} e^{c} = \hat{T}_0 e^{-k^2 K t}$$

Exponential damping of wave coeffts, on time scale $\tau = \dfrac{1}{K k^2} \xrightarrow[k \to \infty]{} 0$

* curvature operator, envy operator, diffusion operator...

## B Behaviour of a thermometer




sfc area $A = 2\pi r \ell + 2\pi r^2$

heat capac. $C = \pi r^2 \ell \rho c$

$m^3$ $kg\, m^{-3}$ $J\, K^{-1} kg^{-1}$

Ohm's Law model

$$Q_H \left[W m^{-2}\right] = \rho_a c_{pa} \frac{T - T_a}{r_H}$$

where $[r_H] = s\, m^{-1}$ and $r_H \propto U^{-1/2}$

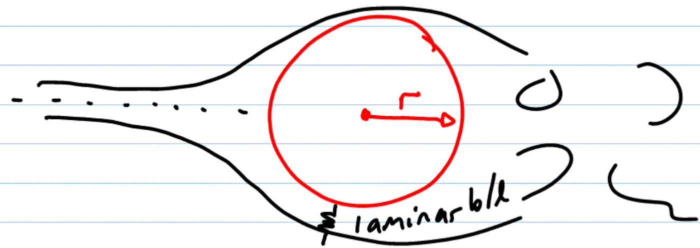
$$= h\,(T - T_a)$$

$$Q^* = K\downarrow - K\uparrow + L\downarrow - L\uparrow$$

Heat balance of the device

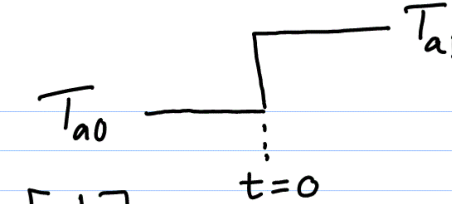
$$C \frac{dT}{dt} = A\left[Q^* - Q_H\right]$$

$\frac{J}{K}$ $\frac{K}{s}$




(i) At steady state $Q^* = Q_H = \frac{\rho_a c_{pa}}{r_H}(T - T_a)$

Radiation error is $T - T_a = \frac{r_H Q^*}{\rho_a c_{pa}}$




(ii) Step response $\quad \frac{dT}{dt} = -N(T - T_a)$

where $N = \frac{A \rho_a c_{pa}}{C r_H} \left[\frac{1}{s}\right]$

$\frac{dT}{dt} + NT = NT_a$ (linear d.e.)

Auxiliary eqn $\quad \frac{dT}{dt} + NT = 0$

$\frac{1}{T}\frac{dT}{dt} = \frac{d \ln T}{dt} = -N$

$\ln T = -Nt + c$

$T = e^{c} e^{-Nt}$

Add P.I. $\quad T = e^{c} e^{-Nt} + d$

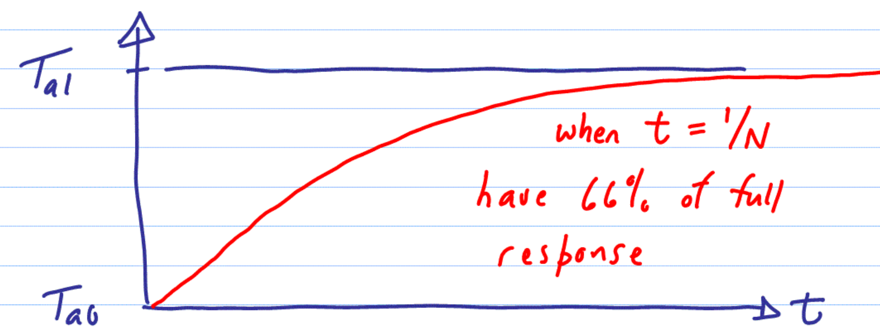
(a) $t = 0 \quad T = T_{a0} = e^{c} + d$

(b) $t \to \infty \quad T = T_{a1} = d$

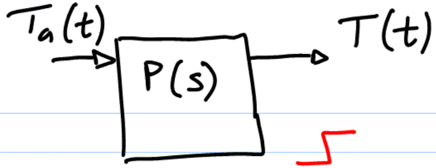
$T = T_{a1} + (T_{a0} - T_{a1}) e^{-Nt}$

$$T = T_{a0} e^{-Nt} + T_{a1}\left(1 - e^{-Nt}\right)$$

timescale $\tau = 1/N$



## (B iii) Sinusoidal Response




Transfer function $P(s) = \dfrac{LT(\text{output})}{LT(\text{input})}$

Let $T_a = \sin \omega t$

$$L[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

$$L\left[\frac{dT}{dt}\right] = s\bar{T} - \underbrace{T(0^+)}_{\text{zero in our case}}$$

Our d.e. is $\dfrac{dT}{dt} + NT = N T_a(t)$

Take L.T. $s\bar{T}(s) + N\bar{T} = N\bar{T}_a$ $\quad \therefore \; P(s) = \dfrac{\bar{T}}{\bar{T}_a} = \dfrac{N}{N+s}$

Power Gain $G(\omega) = P(j\omega)\,P(-j\omega) = \dfrac{N}{N+j\omega}\;\dfrac{N}{N-j\omega}$

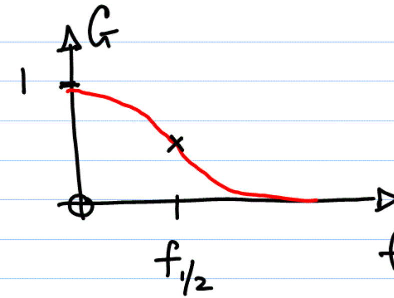
$$= \frac{N^2}{N^2 + \omega^2}$$

$$= \frac{1}{1 + (\omega/N)^2} = \frac{1}{1 + \left[f/(N/2\pi)\right]^2}$$

$$f_{1/2} = \frac{N}{2\pi} = \frac{1}{2\pi\tau}$$

(c) We have figured out the "spectral response" of the device

Recall $G(\omega) = \dfrac{(\text{ampl. out})^2}{(\text{ampl. in})^2}$




Power spectral density (or "spectrum") of a stochastic signal; temperature as example.

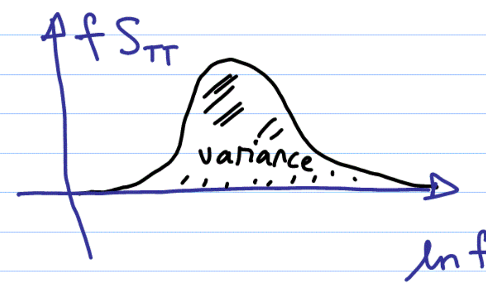
Reynolds decomposition $T_a = \langle T_a \rangle + T_a'$

Variance of $T_a$ is $\sigma_{T_a}^2 = \langle (T_a')^2 \rangle$ = mean square deviation from the mean

$$= \int_0^\infty S_{T_aT_a}(f)\, df$$

spectral density

$$= \int_{-\infty}^{\infty} f\, S_{T_aT_a}(f)\, d\ln f$$

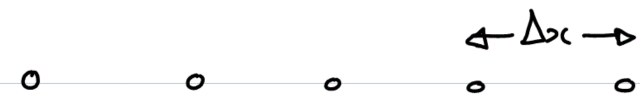



Our thermometer measures

$$\sigma_T^2 = \int_0^\infty G(f)\, S_{T_aT_a}(f)\, df$$

$$\leq \sigma_{T_a}^2$$

(c) Aliasing. Take a uniform grid




This can only represent waves whose

wavelength $\lambda \ge \lambda_{min} = 2\Delta x$ Shannon's sampling theorem

wavenumber $k \le k_{max} = 2\pi/\lambda_{min} = \pi/\Delta x$

Let $x(I) = I\Delta x$, try to represent signal

$$\phi = \sin\Big[\underbrace{k_{max}(1+\varepsilon)}_{k}\, x\Big] = \sin\Big[\frac{\pi}{\cancel{\Delta x}}(1+\varepsilon)\, I\, \cancel{\Delta x}\Big]$$

$$= \sin(\pi I + \pi\varepsilon I) = \underbrace{\cancel{\sin(\pi I)}}_{\text{zero}}\cos(\pi\varepsilon I) + \cos(\pi I)\sin(\pi\varepsilon I)$$

$$= (-1)^{I} \sin \pi\varepsilon I$$

so the wave is not correctly represented on the grid*

* see diagram on web site

# Non-linear computational instability (NLCI)

The non-linear advection terms are continuously generating wave energy at higher and higher wavenumbers ~ a correct, physical process. But the grid cannot represent waves with $k > \pi / \Delta x$ ... so that "wave energy" gets misrepresented (aliased) to lower wavenumbers.