EAS 471, Atmospheric Modelling

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Contents

Chapter 6

Alternative Discretization Methods

Our problems and exercises so far have entailed difference (discretisation/neighbour) equations obtained by the finite difference method. In the simulations of incompressible flow often seen in the engineering literature, *Control Volume* methods are increasingly popular. The present generation of medium range forecast models typically use finite differences on the time and vertical axes, but sometimes use a *Spectral scheme* in the horizontal[1].

6.1 Control Volume Method

Patankar calls the control-volume method a "pre-calculus expression of conservation." Space is divided into adjoining boxes, or control volumes (cv). The control volumes are imaginary – they are invisible to the flow. The governing equations are (analytically) integrated for application to a control volume: the result is a formulation in terms of (i) fluxes (convective and diffusive) across the cv walls (ii) production/destruction within the cv (iii) changes in storage within the cv.

I'll illustrate by looking at a diffusion problem involving the spread of a passive tracer, concentration c [kg m^{-3}]. We'll consider the flow to be laminar and known (the method generalises easily to calculate dispersion in turbulent flow) and the concentration field to be steady, and uniform along the crosswind (y) axis (stationary, 2-d concentration field).

The governing mass conservation equation is

$$\frac{\partial c}{\partial t} = 0 = -\nabla \cdot (\vec{u}\, c - D\, \nabla c) + Q \tag{6.1}$$

where Q [kg m^{-3} s^{-1}] is the volumetric source/sink term. More explicitly, and assuming we limit to two space dimensions,

$$\frac{\partial}{\partial x}\left(u\, c - D\frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial z}\left(w\, c - D\,\frac{\partial c}{\partial z}\right) = Q\,. \tag{6.2}$$

[1]Spectral and Finite Element methods of discretization not covered this (2010) year.

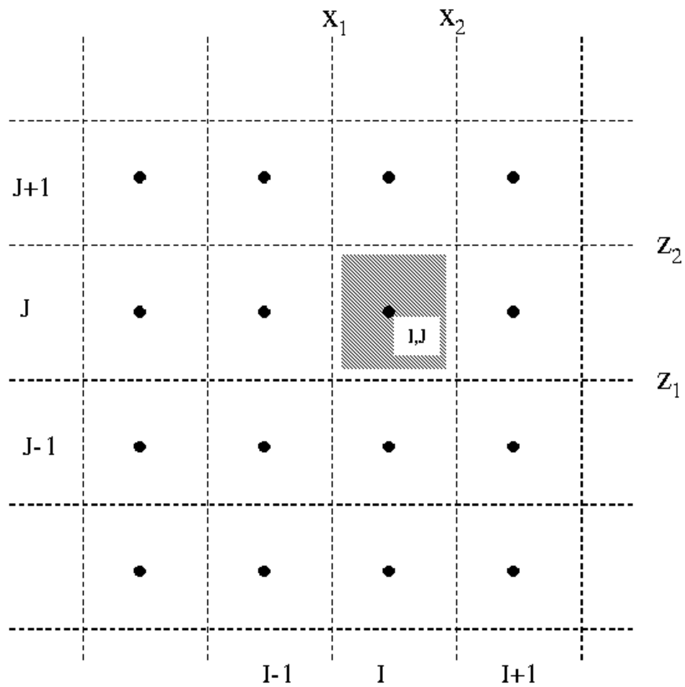

Figure 6.1: Gridpoints on a uniform mesh, and centred within their control volumes. The dashed lines show control volumes faces, eg. x_1, x_2 are the yz-planes defining where faces of the IJ$^{\text{th}}$ control volume cut the x-axis.

Now divide up the space into control volumes[2] as in Figure (6.1). Integrating the differential equation for application within such a volume, we have

$$\int_{x_1}^{x_2} \int_{z_1}^{z_2} \left[\frac{\partial}{\partial x} \left(u\,c - \,D\,\frac{\partial c}{\partial x} \right) + \frac{\partial}{\partial z} \left(w\,c - \,D\,\frac{\partial c}{\partial z} \right) - \,Q \right] dx\,dz = \,0 \tag{6.3}$$

which yields

$$\Delta z\,\left[u\,c - \,D\,\frac{\partial c}{\partial x} \right]_{x_1}^{x_2} + \,\Delta x\,\left[w\,c - \,D\,\frac{\partial c}{\partial z} \right]_{z_1}^{z_2} + \,Q\,\Delta x\,\Delta z = \,0 \tag{6.4}$$

where $\Delta x, \Delta z$ are the face lengths and $[..]_1^2$ denotes a difference in the argument between location 2 and location 1.

Now we clearly have an equation involving fluxes (convective plus diffusive) across faces – ie. exchanges among control volumes. To progress, we have to express the fluxes across the faces in terms of our resolved values of the concentration.

For simplicity let's make all the grid intervals $\Delta x = \Delta z = \Delta$ equal. Let $c(I, J)$ be the concentration and $u(I, J)$ the alongstream velocity at a gridpoint in the middle of the IJ control

[2]For the sake of simplicity we will consider an "unstaggered grid", ie. a grid at which all dependent variables are co-located at the same gridpoints. This is to be contrasted with the "staggered grid" on which (eg.) pressure gridpoints are offset relative to velocity gridpoints.

volume. Then using an upstream scheme for convection, we could estimate:

$$[\,u\,c]_{x_1}^{x_2} = \frac{u(I,J) + u(I+1,J)}{2}\,c(I,J) - \frac{u(I-1,J) + u(I,J)}{2}\,c(I-1,J) \tag{6.5}$$

where we have assumed a linear variation of velocity between gridpoints and thus simply averaged. The diffusion term could be written:

$$\left[\,D\,\frac{\partial c}{\partial x}\,\right]_{x_1}^{x_2} = D_{x_2}\,\frac{c(I,J+1) - c(I,J)}{\Delta} - D_{x_1}\,\frac{c(I,J) - c(I,J-1)}{\Delta}\,. \tag{6.6}$$

In this way we obtain our discretisation equation. Patankar gives a much better interpolation scheme than I have suggested here for the effect of the combination of advection and diffusion across control volume faces.

Example[3]: Calculation of the mean concentration of a gas released continuously from a ground-level line source lying across the wind. Atmosphere assumed neutrally-stratified, wind and turbulence at a given height depending only on two parameters, the friction velocity u_* and the surface roughness length z_0. The mean wind speed is

$$u(z) = \frac{u_*}{0.4}\,\ln\left(\frac{z}{z_0}\right) \tag{6.7}$$

while the mean vertical component (w in the preceding equations) vanishes. We can neglect diffusion along the direction of the mean wind, since advection is overwhelmingly more important; and the (turbulent) diffusivity along the vertical (z) axis is $D = (k_v/S_c)u_* z$ where the von Karman constant $k_v = 0.4$ and the Schmidt number S_c is subject to some uncertainty but appears to be best given the value $S_c \approx 0.6 - 0.7$.

Our discretization equation reduces to

$$\Delta z\;[u\,c]_{x_1}^{x_2} = \Delta x\,\left[\,D\,\frac{\partial c}{\partial z}\right]_{z_1}^{z_2} \tag{6.8}$$

(incidentally note our problem is elliptic on the z-axis). Using an upstream difference for the streamwise advection term gives

$$[\,u\,c]_{x_1}^{x_2} = u(J)\,[c(I,J) - c(I-1,J)] \tag{6.9}$$

(and ensures us a 1-way x-coordinate, ie. a marching problem in x) while for the vertical transport term we can write

$$\begin{aligned}\left[\,D\,\frac{\partial c}{\partial z}\,\right]_{z_1}^{z_2} &= \frac{k_v u_*}{S_c}\left[z_2\,\frac{c(I,J+1) - c(I,J)}{\Delta z} - z_1\,\frac{c(I,J) - c(I,J-1)}{\Delta z}\right] \\ &= D_n\,\frac{c(I,J+1) - c(I,J)}{\Delta z} - D_s\,\frac{c(I,J) - c(I,J-1)}{\Delta z}\end{aligned} \tag{6.10}$$

[3]Here we anticipate material that follows in later chapters.

where $D_n = k_v\, u_*\, z_2/S_c$ is the diffusivity at z_2 (etc). Now if we collect terms we have

$$\begin{aligned} c(I,J)\,\left[u(J)\Delta z + \frac{\Delta x D_n}{\Delta z} + \frac{\Delta x D_s}{\Delta z}\right] &= c(I,J+1)\,\frac{\Delta x D_n}{\Delta z} \\ &+ c(I,J-1)\,\frac{\Delta x D_s}{\Delta z} \\ &+ c(I-1,J)\,u(J)\Delta z \end{aligned} \tag{6.11}$$

This is clearly a "marching problem" in that if we know our solution at streamwise location $I-1$ then we have a closed set of equations for the column matrix $c(I,J)$, $J = 1,2,3....$ It is an implicit-type computational problem, because our unknown at J is linked to its (unknown) neighbours at $J \pm 1$. Thus, when we compute the solution at streamwise location I (already knowing the solution at $I-1$) we have a problem of form:

$$A_J^C c(I,J) = A_J^N\, c(I,J+1) + A_J^S\, c(I,J-1) + B_{I,J}, \quad J = 1..J_{mx} \tag{6.12}$$

where the neighbour coefficients (ie. the vectors A_J^C, A_J^N, A_J^S) are (in our case) fixed vectors, though the "what lies behind" vector B_J will be different for the step $I-1 \rightarrow I$ than for the step $I-2 \rightarrow I-1$ (etc).

6.2 Bubnov-Galerkin Methods: the Spectral and Finite Element methods

Suppose in a given physical problem the desired solution $\phi(x)$ must satisfy the differential equation

$$L\left[\phi(x)\right] = f(x) \ . \tag{6.13}$$

In the Bubnov-Galerkin methods the solution is expressed as a superposition of linearly independent basis functions $\theta_j(x)$, viz.

$$\phi(x) = \sum_{j=1}^{N} a_j \, \theta_j(x) \ . \tag{6.14}$$

Usually N is finite ("truncated set of basis functions"). If the basis functions satisfy the boundary conditions but not the differential equation, this is called a boundary weighted residual method; if they satisfy the differential equation but not the boundary conditions, this is an internal method.

At any point x the error is

$$e(x) = L\left[\sum_{j=1}^{N} a_j \, \theta_j(x)\right] - f(x) \ . \tag{6.15}$$

The free coefficients a_j are optimised by forcing the covariance of the error with each of the basis functions to vanish, viz.

$$\int e(x) \, \theta_j(x) \, dx = 0 \, , \quad j = 1 ... N \ . \tag{6.16}$$

Note that the above equation may be interpreted as stating that the "inner product" defined by

$$(e, \theta_j) = \int e(x) \, \theta_j(x) \, dx \tag{6.17}$$

must vanish for each j; and thus that by definition the error function $e(x)$ is "orthogonal" to each of the basis functions. Probably a more physically-illuminating interpretation is that the integral (or inner product) is a covariance (average along x of the cross-product of $e(x)$ and the basis function $\theta_j(x)$); by forcing that covariance to vanish, we are forcing the error to lie randomly about each of the basis functions (by suitably shaping the basis functions).

The above can be generalised to multi-dimensional problems.

6.2.1 Finite Element Method

Here the basis functions are only locally non-zero: the $\theta_j(x)$ vanish, except in a limited region about each point of interest (grid point). Typically they are chosen to be "tent" functions (also called "chapeau" functions). These functions, eg. along a single space dimension $\theta_j(x)$, are continuous functions of their argument (here x), vanishing except where

$$\begin{aligned} \theta_j(x) &= \frac{x - x_{j-1}}{\Delta x}, \quad x_{j-1} \le x \le x_j \\ \theta_j(x) &= \frac{x_{j+1} - x_j}{\Delta x}, \quad x_j \le x \le x_{j+1} \end{aligned} \tag{6.18}$$

6.2.2 Spectral Method

The θ_j are chosen to be orthogonal functions defined on the entire x axis. In plane geometry they will often be harmonic functions $e^{j(kx+ly)}$. In modern NWP applications they are usually spherical harmonics, with the time dependence of the solution residing in the coefficients (and finite differencing in the vertical). The usual orthogonal basis functions are

$$Y_{m,n}(\mu,\lambda) = P_{m,n}(\mu)\, e^{j\, m\, \lambda} \tag{6.19}$$

where λ is the longitude and $\mu = \sin\phi$ where ϕ is the latitude. Clearly the $e^{jm\lambda}$ implies sines and cosines to represent the zonal structure. The $P_{m,n}$ represent the meridional structure; they are the associated Legendre functions of the 1st kind. For example a model using pressure p as the vertical coordinate would represent the (height-dependent) velocity as:

$$u(\phi,\lambda,p,t) = \sum_m \sum_n u_{m,n}(p,t)\, Y_{m,n}(\mu,\lambda) \tag{6.20}$$

The essential difference between spectral and grid point models is that the meteorological fields are represented by a finite sum of wave components of different amplitude and wavelength, rather than by values on a regularly-spaced mesh. One of the attractive consequences of this is that no differencing is required (in the horizontal directions); horizontal derivatives can be evaluated analytically, because the dependent variables are represented by continuous functions. In particular, finite difference approximations to the non-linear advection terms are not needed, so the spectral model does not suffer from NLCI (the generation of small scale noise). However this does not mean a spectral model has no truncation error. The truncation error of a spectral model is related to the choice of J, the maximum wavenumber which determines the smallest wave represented by the model. If J is small, only large waves are allowed. In nature, wave-wave interactions generate ever-shorter waves, so a forecast made with a model allowing only a very few long wave components would soon depart from reality. Typical spectral models of the 80's carried about 80 waves; a grid model of corresponding resolution would require 2 gridpoints per each shortest wavelength, and so would have a grid spacing of about $1/2\ 360/80 = 2.25^o$ (about 250 km). During integration of a spectral model, transformations are made back to physical space (as opposed to wavenumber space); the spherical grid array that results is called a Gaussian grid. It is easier to calculate the physics in physical space.

6.2.3 Spectral method for 1D non-linear advection eqn

We want to solve

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{6.21}$$

on $-L \le x \le L$ subject to $u(-L,t) = u(L,t)$.

Choose as basis functions

$$\theta_n = e^{j\, k_n\, x} \tag{6.22}$$

where $k_n = n\pi/L$ and $n = -N, -N+1, ... -1, 0, 1..., N-1, N$. These basis functions automatically force the solution to satisfy the boundary conditions. The inner product will for present purposes be

$$(f, g) \;=\; \int_{-L}^{L} f(x,t)\; g^*(x,t)\; dx \tag{6.23}$$

where the * denotes the complex conjugate. The solution is written:

$$u(x,t) = \sum_{-N}^{N} u_n(t)\; e^{j\; n\pi x/L} \tag{6.24}$$

where $u_n(t)$ is the coefficient of the nth wave, in general a complex number (the u_n carry the time-dependence of u(x,t)). Substituting into the advection equation, the error function is:

$$e(x) = \sum_{n=-N}^{N} \frac{du_n}{dt}\, e^{j\; n\pi x/L} + \sum_{n=-N}^{N} u_n(t)\; e^{j\; n\pi x/L} \sum_{m=-N}^{N} \frac{jm\pi}{L}\, u_m(t)\; e^{j\; m\pi x/L}\; . \tag{6.25}$$

To determine the optimal coefficients $u_n(t)$, multiply the error through by $\theta_p^* = e^{-j\,p\,\pi\,x/L}$ and integrate over the range of x, requiring

$$\int_{-L}^{L} e(x)\; e^{-j\,p\,\pi\,x/L}\; dx = \;0 \qquad\qquad p = -N, ...N\; . \tag{6.26}$$

Then

$$\begin{aligned} 0 \;&=\; \sum_{n=-N}^{N} \frac{du_n}{dt} \int_{-L}^{L} e^{-jp\pi x/L}\; e^{jn\pi x/L}\; dx \\ &+\; \sum_{n=-N}^{N} u_n(t) \sum_{m=-N}^{N} \frac{jm\pi}{L}\, u_m(t) \int_{-L}^{L} e^{jn\pi x/L}\; e^{jm\pi x/L}\; e^{-jp\pi x/L}\; dx\; . \end{aligned} \tag{6.27}$$

But we have an orthogonality rule (Spiegel, Advanced Mathematics, p187)

$$\begin{aligned} &\int_{-L}^{L} e^{jm\pi x/L}\; e^{-jp\pi x/L}\; dx \\ =\;& \int_{-L}^{L} \left(\cos\frac{m\pi x}{L} + \;j\; \sin\frac{m\pi x}{L} \right) \left(\cos\frac{p\pi x}{L} - \;j\; \sin\frac{p\pi x}{L} \right) dx \\ =\;& \;2\; L\; \delta_{m,p} \end{aligned} \tag{6.28}$$

so we have

$$\frac{du_p}{dt} \;=\; -\sum_{n=-N}^{N} \sum_{m=-N}^{N} \frac{jm\pi}{2L^2}\, u_n(t)\; u_m(t)\; R^p_{m,n} \tag{6.29}$$

This is a set of $2N+1$ coupled ordinary differential equations for the advancement in time of the complex coefficients $u_p(t),\; p = -N.....N$. The interaction coefficients

$$R^p_{m,n} = \int_{-L}^{L} e^{j(m+n)\pi x/L}\; e^{-jp\pi x/L}\; dx = \;2\; L\; \delta_{m+n,p} \tag{6.30}$$

vanish unless $m + n = p$, in which case waves m, n are interacting to produce a wavenumber $p = m + n$ component. They may be calculated in advance and stored for all the interactions which will be retained. Splitting the coefficients into their real and imaginary parts,

$$u_n(t) = u_n^R(t) + j u_n^I(t) \tag{6.31}$$

it is straightforward to show that

$$\begin{aligned}
\frac{du_p^R}{dt} &= \sum_{n=-N}^{N} \sum_{m=-N}^{N} \frac{m\pi}{L} \left[u_n^I(t)\, u_m^R(t) \; + u_m^I(t)\, u_n^R(t) \right] \delta_{m+n,p} \\
\frac{du_p^I}{dt} &= \sum_{n=-N}^{N} \sum_{m=-N}^{N} \frac{m\pi}{L} \left[u_n^I(t)\, u_m^I(t) - \; u_n^R(t)\, u_m^R(t) \right] \delta_{m+n,p}
\end{aligned} \tag{6.32}$$

Procedure: We choose a value for N, say $N = 10$ (21 waves). The initial velocity field must now be represented by the decomposition:

$$u(x, 0) = \sum_{n=-10}^{10} u_n(t)\, e^{jn\pi x/L} \tag{6.33}$$

Suppose we initialize with the special case of a single long cosine wave corresponding to $n = 1$, ie. $u(x, 0) = \cos(\pi x/L)$. Then only the coefficient u_1 is non-zero at commencement of the integration (ie. $u_1(0) \neq 0$), with $u_1^R(0) = 1$ and $u_1^I(0) = 0$. But after the first timestep, at time Δt, we will have non-zero $u_2(t)$ due to the interaction $1 + 1 \rightarrow 2$. At $2\Delta t$ we have non-zero $u_3(2\Delta t)$ due to the interaction $1 + 2 \rightarrow 3$. Soon all coefficients will be non-zero, after which the further steepening of the gradients implied by the physics is lost due to the retaining of only the limited set of waves.

The non-linear advection equation is a pathological example. When the spectral procedure is applied to weather prediction, it is assumed that the additional terms in the governing equations in any case prohibit the attainment of indefinitely strong gradients.

Chapter 7

Resolved and Unresolved Scales of Motion

The atmosphere is a continuum. The smallest eddies are very much larger than the molecular mean free path, so one can define each fluid "point" as being actually a small volume containing many molecules and having well-defined pressure, temperature, etc. So, in the atmospheric continuum we clearly never have complete (every-point, every-instant) knowledge of its state. Therefore, in any description of the atmosphere (whether it be concerned with global meteorology, the meso-scale, or micro-meteorology) it is helpful to introduce the notion of "resolved" and "unresolved" scales of motion. We split any variable, say the vertical velocity w, into resolved and unresolved parts, thus:

$$w = \overline{w} + w' \tag{7.1}$$

where $w(x, y, z, t)$ is the total instantaneous vertical velocity, $\overline{w}(x, y, z, t)$ is the resolved motion[1], and w' is the deviation from the resolved motion, the "unresolved motion" or "fluctuation." We then declare that we are concerned to predict/describe the "resolved fields," $\overline{u}, \overline{v}, \overline{w}, \overline{p}, \overline{T}, \overline{\rho}_v$. But it is an inescapable fact that the unresolved field usually has a non-negligible influence on the evolution of the resolved field.

Now what might this "resolved" field $\overline{w}$ be? In what sense is $\overline{w}$ a "filtered" field? As a first example, let us suppose we are concerned with the "synoptic scale," ie. working with dependent variables which have been averaged in the horizontal plane (or on a pressure surface) over distances (X, Y) large enough to average out microscale and mesoscale variations: eg.

$$\overline{w} = \frac{1}{XY} \int_{-X/2}^{X/2} \int_{-Y/2}^{Y/2} w(x, y, z)\, dx\, dy \tag{7.2}$$

Note that we could and probably should have also averaged in the vertical, since we cannot resolve all detail along any space dimension.

This averaging operation smooths out the sharp updrafts and downdrafts about a range of hills, or the updrafts about a sharp lake-land or ocean-land boundary, or the intense updrafts within

[1] Common alternative notations for the resolved field are the upper case, W and the angle-bracket $< w >$.

individual cumulus clouds: and we are left with a spatially-averaged vertical velocity $\overline{w}$ which is small. Note that $\overline{w}$ remains a function of x, y, but it is a much smoother function of x, y than w. Detail (unresolved structure) has been "filtered" out.

Now, does the existence of an *unresolved* field w' have any effect on the evolution of $\overline{w}$? Yes. For example, a population of cumulus, though each too small to be a resolved feature (cloud width $L << X, Y$), collectively causes very efficient vertical heat and vapour transport, that results in a vertical heat flux by the "unresolved flow" that may even exceed the resolved heat flux $\overline{Q}_H = \rho\, c_p \overline{w}\, \overline{T}$.

We have established that in meteorology/climatology and in oceanography, no matter what the scale under consideration, one should always commence by settling the issue of scales to be resolved. That done, we need to find "evolution equations" (prognostic or governing equations) for the resolved field $\overline{u}$ etc., ie. we want differential equations for $\partial \overline{u}/\partial t$, etc. And we must show, formally and quantitatively, the way in which the unresolved field affects the evolution of the resolved field. This is straightforward. We simply average the governing equations (a procedure pioneered by O. Reynolds).

7.1 Averaging the Governing Equations

Let f and g be arbitrary flow variables. Then:

$$\begin{aligned} f &= \overline{f} + f' \\ g &= \overline{g} + g' \end{aligned} \tag{7.3}$$

We would like the averaging process to satisfy the following four conditions:

$$\begin{aligned} \overline{f+g} &= \overline{f} + \overline{g} \\ \overline{\alpha\, f} &= \alpha\, \overline{f} \\ \overline{\frac{\partial f}{\partial s}} &= \frac{\partial \overline{f}}{\partial s} \\ \overline{\overline{f}\, g} &= \overline{f}\, \overline{g} \end{aligned} \tag{7.4}$$

where α is (any) constant and $s = x_1, x_2, x_3$ or t.

If these four requirements are met, then in addition the following properties hold true (prove for yourselves):

$$\begin{aligned} \overline{\overline{f}} &= \overline{f} \\ \overline{f'} &= 0 \\ \overline{\overline{f}\, \overline{g}} &= \overline{f}\, \overline{g} \\ \overline{\overline{f}\, g'} &= 0 \end{aligned} \tag{7.5}$$

We will average the equations as if our averaging operation was ideal, but in reality neither a space- nor a time-average exactly satisfies rules 3, 4. Anthes (1977) is more careful about this.

7.1.1 Example: Averaging the Vapour Conservation Equation

Conservation of water vapour is expressed by:

$$\frac{\partial \rho_v}{\partial t} = -\nabla \cdot \vec{F} + Q_v \tag{7.6}$$

where $\vec{F} \equiv F_j$ is the (total) vector flux density of water vapour, given by

$$F_j = u_j\, \rho_v - \rho\, D_v\, \nabla \frac{\rho_v}{\rho}\,. \tag{7.7}$$

We shall neglect the contribution of molecular diffusion to transport of water vapour.

Now noting that averaging commutes with differentiation, we have

$$\overline{\frac{\partial \rho_v}{\partial t}} = \frac{\partial \overline{\rho}_v}{\partial t}$$

and

$$\overline{\frac{\partial F_j}{\partial x_j}} = \frac{\partial \overline{F}_j}{\partial x_j}$$

so the average form of the vapour conservation equation is:

$$\frac{\partial \overline{\rho}_v}{\partial t} = -\frac{\partial \overline{F}_j}{\partial x_j} + \overline{Q_v} \tag{7.8}$$

where

$$\overline{F}_j = \overline{u_j\, \rho_v} \tag{7.9}$$

Now substitute $u_j = \overline{u}_j + u_j'$ and multiply to get

$$\begin{aligned} \overline{F}_j &= \overline{\left(\overline{u}_j + u_j'\right)\ \left(\overline{\rho}_v + \rho_v'\right)} \\ &= \overline{u}_j\, \overline{\rho}_v\ +\ \overline{u_j'\, \rho_v'}\,. \end{aligned} \tag{7.10}$$

We will temporarily neglect the source term Q_v (though it can easily be put back in, and needs to be to account for evaporation/condensation with respect to any liquid water present). Then, the averaged form of the equation, ie. the evolution equation for $\overline{\rho}_v$ is:

$$\begin{aligned} \frac{\partial \overline{\rho}_v}{\partial t} = &- \frac{\partial}{\partial x}\, \overline{u}\, \overline{\rho}_v - \frac{\partial}{\partial y}\, \overline{v}\, \overline{\rho}_v - \frac{\partial}{\partial z}\, \overline{w}\, \overline{\rho}_v \\ &- \frac{\partial}{\partial x}\, \overline{u'\rho_v'} - \frac{\partial}{\partial y}\, \overline{v'\rho_v'} - \frac{\partial}{\partial z}\, \overline{w'\rho_v'} \end{aligned} \tag{7.11}$$

The first three terms on the rhs involve the resolved variables (and their gradients) and are simply "resolved advection," or, advection by the resolved flow, expressed in transport form. It is the second three terms that are novel. They are new unknowns, and represent the divergence of fluxes of water vapour carried by the unresolved flow. That is to say, $\overline{w'\, \rho_v'}$ is a convective vertical flux density of water vapour, carried by the unresolved flow, and so on. And if $\overline{w'\, \rho_v'}$ changes with height, there results an influence on the resolved humidity field $\overline{\rho}_v$. Whereas the governing equation for ρ_v is "closed" if we regard the velocity field and Q_v as known (one equation in one unknown, ρ_v), the equation we have derived for the evolution of the resolved humidity is unclosed, for even if we know $\overline{u}, \overline{v}, \overline{w}, \overline{Q_v}$, there appear in our equation in addition to $\overline{\rho}_v$ the unknown unresolved fluxes. This proliferation of unknowns is called "the closure problem."

So, the form of the influence of unresolved flow upon the resolved flow is that there arise fluxes carried by the unresolved flow, and the divergences of those fluxes appear as a "forcing" in the evolution equation for the resolved flow. It is quite common in numerical weather prediction to make an artificial separation into "dynamics" and "physics" in the following manner:

$$\frac{\partial \overline{\rho}_v}{\partial t} = \left(\frac{\partial \overline{\rho}_v}{\partial t}\right)_{dyn} + \left(\frac{\partial \overline{\rho}_v}{\partial t}\right)_{phys} \tag{7.12}$$

where

$$\begin{aligned} \left(\frac{\partial \overline{\rho}_v}{\partial t}\right)_{dyn} &= -\frac{\partial}{\partial x}\,\overline{u}\,\overline{\rho}_v - \frac{\partial}{\partial y}\,\overline{v}\,\overline{\rho}_v - \frac{\partial}{\partial z}\,\overline{w}\,\overline{\rho}_v \\ \left(\frac{\partial \overline{\rho}_v}{\partial t}\right)_{phys} &= -\frac{\partial}{\partial x}\,\overline{u'\rho_v'} - \frac{\partial}{\partial y}\,\overline{v'\rho_v'} - \frac{\partial}{\partial z}\,\overline{w'\rho_v'} + Q_v \end{aligned} \tag{7.13}$$

By the same method one may show that there arise unresolved horizontal and vertical heat fluxes $\rho c_p \left(\overline{u'T'},\ \overline{v'T'},\ \overline{w'T'}\right)$, whose divergences appear in the evolution equation for mean temperature $\overline{T}$, and so the tendency in $\overline{T}$ is likewise split into dynamics and physics, with

$$\left(\frac{\partial \overline{T}}{\partial t}\right)_{phys} = -\frac{\partial}{\partial x}\,\overline{u'T'} - \frac{\partial}{\partial y}\,\overline{v'T'} - \frac{\partial}{\partial z}\,\overline{w'T'} + Q_T \tag{7.14}$$

where the source term Q_T includes any latent heat addition/removal and the divergence of the radiative heat flux

$$Q_T = -L\,Q_v - \nabla \cdot \vec{R}^* \tag{7.15}$$

where L is the latent heat of vapourisation/sublimation. Note that if Q_v is positive, vapour is being created by evaporation, which consumes thermodynamic energy and makes a negative contribution to Q_T.

Completing this, the "physics" term in the $\overline{u}$-momentum equation, often simply called "friction," is:

$$\left(\frac{\partial \overline{u}}{\partial t}\right)_{phys} = -\frac{\partial \overline{u'u'}}{\partial x} - \frac{\partial \overline{v'u'}}{\partial y} - \frac{\partial \overline{w'u'}}{\partial z} \tag{7.16}$$

where $\tau_{xz}/\rho = \overline{u'w'}$ is a momentum flux (per unit mass) carried by the unresolved flow, sometimes called a "Reynolds stress."

We have now covered the decomposition into resolved and unresolved scales, a matter that needs to be considered prior to (or in the context of) virtually any modelling exercise in meteorology or oceanography. We have seen that the fluxes carried by the unresolved motion feed back on the resolved fields of motion and temperature and humidity, and that the evolution equation for the resolved field is unclosed, until we specify (by some means) the magnitudes of the unresolved fluxes (the closure problem).

Chapter 8

The horizontally-homogeneous atmospheric surface layer

It is envisaged that one or more of the assignments will be a realistic simulation of the dispersion of a gas (or particles) from a point source in the atmospheric surface layer, roughly the lowest 50-100 m of the atmosphere. Hence it will be useful to divert into the *micro-meteorology* of this layer, in its simplest guise (most idealized form).

For now we will define the surface layer ("ASL") to be a shallow layer adjacent to ground, with depth z_s of order $\delta/10$ where δ is the depth[1] of the entire Atmospheric Boundary Layer (ABL, also known as the 'friction layer'); in this ASL the vertical gradients in mean windspeed, temperature, and humidity are very strong, and substantial (turbulent, convective) fluxes of heat, momentum and moisture pass upward or downward, linking the surface to the atmosphere. Since the ASL is shallow, these vertical fluxes of heat, mass and momentum change only by a small fraction between ground and z_s, so sometimes the surface layer is called the 'constant flux layer'.

In the ABL the velocity field is (to a satisfactory level of approximation) non-divergent, and this is true of the instantaneous field, fluctuation field, and mean field. Now the mean velocity field being non-divergent, it follows at once from

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0\,, \quad \text{(continuity)} \tag{8.1}$$

$$\overline{w}(0) = 0 \quad \text{(boundary condition)} \tag{8.2}$$

that *if* flow statistics are independent of x, y *then* (for all z) $\overline{w}(z) = 0$. We can make $\overline{v} = 0$ at ground too, by choice of the coordinate orientation, but due to turning of the mean wind with height (due to the Coriolis force), $\overline{v}$ will not necessarily vanish aloft (though this turning is often neglected in the shallow "surface-layer").

'**Horizontal-homogeneity**' ('horizontally-uniformity') - *statistics* do not vary in the horizontal
'**Stationarity**' ('steady state') - *statistics* do not vary in time

[1] Another common symbol for the depth of the ABL is 'z_i', referring to the height above ground of the base of a capping inversion (hence the 'i') that is a common (but not universal) feature of the daytime ABL.

The above simplifications are perhaps never 100% legitimate, but in adopting them we can at least conceptualize an ideal surface layer, and doing so proves very useful.

8.1 Mean momentum equations

The mean momentum equations, obtained by Reynolds-averaging the Navier-Stokes equations[2], are:

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{u}_i\,\overline{u}_j + \overline{u_i'\,u_j'}\right) = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial x_i} + g\frac{\overline{T}}{T_0}\,\delta_{3i} - F_{ci} \tag{8.3}$$

(F_{ci} denotes the Coriolis force; $\overline{p}$ and $\overline{T}$ are the pressure and temperature *departures* from a hydrostatic and asiabatic reference state; I have neglected viscous momentum transport). Assuming horizontal homogeneity the horizontal momentum equations reduce to

$$\frac{\partial \overline{u}}{\partial t} = \frac{-1}{\rho_0}\frac{\partial \overline{p}}{\partial x} + f\,\overline{v} - \frac{\overline{u'w'}}{\partial z}\,, \tag{8.4}$$

$$\frac{\partial \overline{v}}{\partial t} = \frac{-1}{\rho_0}\frac{\partial \overline{p}}{\partial y} - f\,\overline{u} - \frac{\overline{v'w'}}{\partial z}\,, \tag{8.5}$$

where f is the Coriolis parameter ($f = 2\Omega\sin\phi$, where ϕ is latitude). These equations control the mean wind profiles in the ABL.

Now if we make the further simplification of assuming stationarity, the streamwise component is

$$\frac{\partial\overline{u'w'}}{\partial z} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial x} + f\overline{v} \tag{8.6}$$

(e.g. Garratt, 1992, eqn 2.48). We have a balance of the pressure gradient, Coriolis and 'friction' forces, where 'friction' is expressed as the divergence of the turbulent momentum flux. We can orient our surface layer coordinate system so that $\overline{v} = 0$. In addition, we already know from continuity equation that $\overline{w} = 0$. So we have a single non-zero component of the mean velocity, $\overline{u}(z)$. Unfortunately this reprehensible actor refuses to appear in its own governing equation. Also, our equation contradicts the notion of the momentum flux being height independent — it can't be, not if Eq. (8.6) is true. Our 'constant stress layer' is a convenient fiction.

Later we'll see how a theory for $\overline{u}(z)$ is developed, either by an intuitive scaling approach (MOST), or by heuristic introduction of a flux-gradient closure.

[2]Under the Boussinesq approximation, which permits treating the velocity field as incompressible and the density as constant except where multiplied by gravity.

8.2 TKE budget assuming stationarity and horizontal uniformity

The existence of a deep friction layer is a consequence of vertical momentum transport[3] by turbulent vertical velocity fluctuations. In a turbulent region of the atmosphere, by definition, the turbulent kinetic energy (strictly, per unit mass)

$$k = \frac{1}{2}\left({\sigma_u}^2 + {\sigma_v}^2 + {\sigma_w}^2\right) \tag{8.7}$$

is non-zero (k is commonly used as a diagnostic of the turbulence, i.e. it is one of the key statistics). Turbulence dissipates kinetic energy, so unless supplied with energy it will die out. By understanding the 'turbulent kinetic energy budget' we can understand why the depth of the layer within which there is turbulence (causing momentum transfer, heat transfer etc.) undergoes diurnal and seasonal cycles[4].

In slightly simplified form the TKE budget for the horizontally-uniform ABL is

$$\begin{aligned}\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} = 0 \quad &= \quad -\overline{u'w'}\,\frac{\partial \overline{u}}{\partial z} \; - \overline{v'w'}\,\frac{\partial \overline{v}}{\partial z} + \frac{g}{T_0}\,\overline{w'T'} \\ &- \quad \epsilon \; + (\text{term often modelled as diffusion})\end{aligned} \tag{8.8}$$

(by choice of our coordinate system it is possible in the *surface layer* to eliminate the term involving $\overline{v}$).

The first two terms on the rhs are conventionally called "shear production". The stresses due to unresolved scales work on the mean flow, converting MKE (mean KE) to TKE. The third term on the rhs is buoyant production, the mean rate of working (force x velocity) by the buoyancy force $\overline{(g\ T'/T_0)\ w'}$. The term represented by ϵ is the turbulent kinetic energy dissipation rate, and if written explicitly it involves viscosity. It is often modelled as $\epsilon = k/\tau$ where τ is a timescale. The terms indicated as 'modelled as diffusion' comprise a transport term, made up of two parts, pressure transport and turbulent transport.

8.3 Neutral wind profile for the surface layer via K-theory

What does the (putative - or rather, approximate) constancy of $\overline{u'w'}$ tell us about the mean wind, $\overline{u}(z)$? Nothing. So far, we only know that $\overline{u} = 0$ at ground; and we know it increases aloft.

In a laboratory wall shear layer the Coriolis force is negligible and consequently there is no turning of wind direction of height. A suitable empirical formula for the mean velocity as function of height is

$$\overline{u}(z) = U_\delta\ \left(\frac{z}{\delta}\right)^m \tag{8.9}$$

[3]If you like, of momentum *deficit*. The no-slip condition at ground means there is a streamwise velocity deficit at that level exactly equal to the Geostrophic (or free stream) windspeed; that velocity deficit is mixed aloft by the fluctuating vertical motion.

[4]The TKE budget also offers many other crucial insights into the fluid dynamics of turbulence.

where U_δ is the free stream velocity (velocity outside the boundary layer) and m is an empirical parameter. Suitably tuned, this can be made to fit the mean windspeed or perhaps even mean velocity over a restricted region of the ABL as well, typically by taking a reference height H that might be of order of the surface layer depth z_S, viz.

$$\overline{u}(z) = U_H \left(\frac{z}{H}\right)^m \tag{8.10}$$

But this is pure empiricism, and not very physical. To obtain a more satisfying and insightful formula for $\overline{u}(z)$ for the neutrally-stratified and horizontally-uniform surface layer, we will turn to the closure assumption (which we shall tentatively assume useful), which relates the velocity *gradient* to the shear stress:

$$\overline{u'w'} = -K_m \left(\frac{\partial \overline{u}}{\partial z} + \frac{\partial \overline{w}}{\partial x}\right) \tag{8.11}$$

(correct symmetry demands the term involving $\overline{w}$, but we know it vanishes in our hhASL).

K-theory was introduced by the pioneers as a model for the turbulent convective fluxes, which they recognized as needing to be known but were unable to measure. K-theory represents turbulent convection as if it were a diffusive process, ie. models turbulent convection by analogy to molecular diffusion. It is therefore fundamentally wrong - but it works, in simple situations.

Now the eddy viscosity K_m [m^2 s^{-1}] is (dimensionally) the product of a velocity scale times a length scale. It has (qualitatively) to do with the efficiency of the turbulence (ie. the unresolved scales of motion) in causing down-gradient transfer of momentum. So why not choose σ_w as the velocity scale? No vertical motion - no mixing. Also, $\sigma_w \propto u_*$ and u_* relates to TKE; so we can pick either u_* or σ_w as our scale. It is conventional to pick u_*.

Observations have shown that the turbulence lengthscale[5] (however defined) increases linearly with distance from ground (this is only true quite close to ground: it does not increase indefinitely). This suggests using as our length scale a multiple of z, say $k_v z$ (where k_v is called von Karman's constant). Doing so, we have:

$$-\,{u_*}^2 = -k_v\, z\, u_* \frac{\partial \overline{u}}{\partial z} \tag{8.12}$$

or rearranging:

$$\frac{\partial \overline{u}}{\partial z} = \frac{u_*}{k_v\, z} \tag{8.13}$$

If we define a constant of integration z_0 (the "surface roughness length") as the height where $\overline{u} = 0$, we can integrate to get the "log wind profile"

$$\overline{u}(z) = \frac{u_*}{k_v}\, \ln\left(\frac{z}{z_0}\right)\ . \tag{8.14}$$

Experiments show that with $k_v = 0.4$ this is a good description of the mean wind in the undisturbed surface-layer under neutral stratification, provided we are not too close to ground ($z \gg z_0$). This

[5]This may be defined formally. For example one may take as a time scale the reciprocal of the frequency at which the power spectrum peaks, then multiply by the velocity standard deviation. The length scale is considered to characterize the size of the large, energetic eddies - not the isotropic small fry.

of course prohibits use of the log profile within and close above crops. We will later see how the Monin-Obukhov similarity theory generalizes our finding for the stratified (but still horizontally-uniform) surface layer, without need to introduce K-theory.

By measuring the mean wind $\overline{u}$ at two (but preferably several) heights we can deduce the friction velocity from the slope:

$$\frac{\Delta \overline{u}}{\Delta \ln(z)} = \frac{u_*}{k_v} \tag{8.15}$$

(and we can obtain z_0 from the $\overline{u} = 0$ intercept). Thus, bearing in mind that once we know u_* we can infer many other properties of the surface layer (surface drag τ, turbulence TKE, σ_w, etc.), a very great deal of information can be obtained simply by measuring the mean wind profile.

Remember: the log law is severely limited in its validity.

8.4 Monin-Obukhov Similarity Theory (MOST)

Why do we need a similarity theory? Because the governing equations for the mean flow are unclosed, and even when closure assumptions are supplied, they cannot be solved except numerically.

The MO theory is very simple, and has proven very successful. Its use is now standard in many fields, including surface layer treatment in weather and climate models. Note though, that (strictly) MOST is valid only in *horizontally-uniform* conditions, and only at heights $z \gg z_0$.

Monin and Obukhov assumed there is a layer far enough above ground that scales of surface roughness do not affect the flow, yet close enough to ground that the depth δ of the ABL does not affect the flow. Within this layer $z_0 \ll z \ll \delta$ it is assumed that the turbulence is "controlled" by:

- the kinematic heat flux through the surface layer $\overline{w'T'} = Q_H/(\rho c_p)$
- the friction velocity u_*, defined in terms of the magnitude of the kinematic momentum flux through the surface layer

$$u_* = \left[\left(\overline{u'w'}\right)^2 + \left(\overline{v'w'}\right)^2 \right]^{1/4} \tag{8.16}$$

- a buoyancy parameter g/T_0, where T_0 is the mean (Kelvin) temperature

From the dimensional vantage point (see Appendix), this suggests using these scales for non-dimensionalisation:

$$\begin{aligned} & u_* \\ T_* &= -\frac{Q_H}{\rho c_p\, u_*} \\ L &= \frac{-{u_*}^3}{k_v\, \frac{g}{T_0}\, \frac{Q_H}{\rho\, c_p}} \end{aligned} \tag{8.17}$$

(note that only two of the set u_*, T^*, L are independent). The von Karman constant k_v is included in definition of L simply by convention. Note that T_0 would have been an unhelpful choice for a temperature scale: we want to quantify the influence of stratification, ie. of a heat flux through the surface layer. T_0 contains no information on that flux: a surface layer having $T_0 = 300^o K$ could be stable, neutral, or unstable!

With this choice of scales, for the wind shear we might write:

$$\frac{k_v z}{u_*} \; \frac{\partial \overline{u}}{\partial z} = \phi_m \left(\frac{z}{L} \right) \tag{8.18}$$

where $\phi_m()$ is some unknown "universal function." Of course $k_v z$ might equally as well be replaced by L, in which case we should have a different function on the rhs, ϕ^*.

This probably seems arbitrary. A systematic derivation of Monin-Obukhov scaling is easy, but let's just say that MOST is justified by its utility. Its simple, and it works fairly well.

8.5 MO Surface-Layer Profiles

Now the purpose of the similarity theory is to find compact and useful formulae for the properties of the surface-layer: and properties of dominant interest are the profiles of the mean wind $\overline{u}$, temperature $\overline{T}$, and humidity ρ_v. Since scales pertaining to the ground have not been included, we will work with the gradients in these properties. We write:

$$\begin{aligned}
\frac{k_v z}{u_*} \; \frac{\partial \overline{u}}{\partial z} &= \phi_m \left(\frac{z}{L} \right) \\
\frac{k_{vh} z}{T_*} \; \frac{\partial \overline{T}}{\partial z} &= \phi_h \left(\frac{z}{L} \right) \\
\frac{k_{vw} z}{\rho_{v*}} \; \frac{\partial \overline{\rho_v}}{\partial z} &= \phi_w \left(\frac{z}{L} \right) \\
\frac{k_{vc} z}{c_*} \; \frac{\partial \overline{c}}{\partial z} &= \phi_c \left(\frac{z}{L} \right)
\end{aligned} \tag{8.19}$$

Here $k_v z$ (etc.) could with equal validity be replaced by z or by L. This merely changes the numerical values of the dimensionless Monin-Obukhov universal similarity functions which appear on the rhs. The argument of these functions is the dimensionless height z/L. We begin by intentionally distinguishing the various von Karman constants, ie. $k_v, k_{vh}, k_{vw}, k_{vc}$, in order to require that the universal functions $\phi_m, \phi_h, \phi_w, \phi_c$ take unit value in the neutral limit, $z/L \to 0$.

But there is one aspect of the above expressions that needs to be modified. The mean temperature profile that corresponds to the state of zero heat flux (ie. neutrality, $T_* = 0$) is not $\partial \overline{T}/\partial z = 0$ but

$$\frac{\partial \overline{T}}{\partial z} = \Gamma \tag{8.20}$$

where $\Gamma = \frac{g}{c_p} \approx 0.01$ [K m^{-1}] is the "adiabatic lapse rate". Alternatively, of the neutrally-stratified layer we may say that the mean potential temperature $\overline{\theta}$ is uniform,

$$\frac{\partial \overline{\theta}}{\partial z} = 0 \tag{8.21}$$

The ASL is sufficiently deep that we need to re-write our equation for the temperature profile as:

$$\frac{k_{vh} z}{T_*} \left(\frac{\partial \overline{T}}{\partial z} - \Gamma \right) = \phi_h \left(\frac{z}{L} \right) \tag{8.22}$$

or better as

$$\frac{k_{vh} z}{T_*} \frac{\partial \overline{\theta}}{\partial z} = \phi_h \left(\frac{z}{L} \right) \tag{8.23}$$

Now we have expressions for the gradients in wind, temperature, humidity. This doesn't get us far till we know the universal dimensionless (ϕ) functions. These have been the subject of a handful of major micrometeorological "flux-gradient" experiments (so called because they established the relationships between fluxes and gradients) in the past three decades, and are now well-established. Before looking at the functions, a word about the Monin-Obukhov length L, and about the concept of eddy diffusivity (which has not arisen so far in this similarity analysis).

8.6 About the Obukhov Length

Loosely, L is a "stability parameter," positive in stable stratification, negative in unstable stratification, and infinite (+ or -) under neutral stratification. L is sometimes called the "height of the substrate of dynamic turbulence," because its magnitude indicates (speaking qualitatively) the depth of the layer in which shear production $-\overline{u'w'}\, \partial \overline{u}/\partial z$ of TKE is more important in the TKE balance than buoyant production/destruction $g/T_0\ \overline{w'T'}$. The ratio of these terms in the TKE budget is by definition the flux Richardson number and the flux-gradient experiments have shown that (in the case of unstable stratification):

$$R_i{}^f = \frac{\frac{g}{T_0}\ \overline{w'T'}}{\overline{u'w'}\ \frac{\partial \overline{u}}{\partial z}} \approx \frac{z}{L} \tag{8.24}$$

8.7 Exchange coefficients

The MO similarity theory does not depend on the introduction of K-theory. However if we wish to introduce the eddy viscosity (eg.), we have by definition:

$$\overline{u'w'} = -K_m \frac{\partial \overline{u}}{\partial z} \tag{8.25}$$

and comparison with the above relation for the wind shear shows that we must have:

$$K_m = \frac{k_v u_* z}{\phi_m \left(\frac{z}{L} \right)} \tag{8.26}$$

Similarly,

$$\begin{aligned} K_h &= \frac{k_v u_* z}{\phi_h \left(\frac{z}{L} \right)} \\ K_w &= \frac{k_v u_* z}{\phi_w \left(\frac{z}{L} \right)} \end{aligned} \tag{8.27}$$

Thus we are supplied with closure relations, but these are valid only under the conditions such that MO similarity is valid: not too close to the "ground" or canopy, in a horizontally-uniform ASL.

8.8 Relationship between Flux and Gradient Richardson Numbers

We've already encountered the flux Richardson number, defined by eqn(8.24). Substituting the Monin-Obukhov expressions for the the wind shear it follows that:

$$R_i{}^f = \frac{z}{L}\,\frac{1}{\phi_m} \tag{8.28}$$

The gradient Richardson number is another dimensionless stability parameter, which can be evaluated from gradients alone:

$$R_i{}^g = \frac{\frac{g}{T_0}\,\frac{\partial \overline{T}}{\partial z}}{\left(\frac{\partial \overline{u}}{\partial z}\right)^2} \tag{8.29}$$

Using the MO expressions for the temperature and velocity gradients, this may be written:

$$R_i{}^g = \frac{z}{L}\frac{\phi_h}{{\phi_m}^2} = \frac{\phi_h}{\phi_m}\,R_i{}^f \tag{8.30}$$

Experimental data are mostly consistent with $\phi_h \approx {\phi_m}^2$.

8.8.1 The ITCE (International Turbulence Comparison Experiment)

Yaglom (1977) gives an excellent discussion of the difficulty of the experiments to determine the MO functions. The desire to determine these ϕ's spurred development of fast-response sensors for direct measurement of the fluxes (eddy correlation). In the days before computers, the need for fast sampling and cross multiplication of signals lead to some very ingenious electronic work, based on analog rather than digital processing (eg. a fascinating glimpse of the concepts and equipment of micro-meteorology in the 1950's is given by Halstead et al., 1957).

The ITCE experiment was lead by CSIRO, with participation by Argonne National Labs and a group from the Soviet Union. The site was flat, open grazing land in New South Wales, with the slope less than 2/10000. Fast response sensors determined all covariances (fluxes) and turbulence statistics, while at the ground drag plates determined drag (τ_0) and a Lysimeter the evaporation rate (E_0). In conjunction with the flux measurements, masts supported cup anemometers and temperature sensors.

The ITCE analysis by Dyer and Bradley (1982) did not presuppose the value of k_v, their starting point being

$$\begin{aligned}
\frac{k_{vm}z}{u_*}\,\frac{\partial \overline{u}}{\partial z} &= \phi_m\left(\frac{z}{L}\right) \\
\frac{k_{vh}z}{T_*}\,\frac{\partial \overline{T}}{\partial z} &= \phi_h\left(\frac{z}{L}\right) \\
\frac{k_{vw}z}{\rho_{v*}}\,\frac{\partial \overline{\rho}_v}{\partial z} &= \phi_w\left(\frac{z}{L}\right)
\end{aligned} \tag{8.31}$$

where $q_* = -E/\rho u_*$, $T_* = -Q_H/\rho c_p u_*$. As noted earlier, allowing different k_v's permits to insist that $\phi_m(0) = \phi_h(0) = \phi_w(0) = 1$ for the neutral limit $z/|L| = 0$. Dyer and Bradley concluded that within the accuracy of the experiment, $k_{vm} = k_{vh} = k_{vw} = 0.4$ (so we can call them all simply k_v) and therefore the eddy diffusivities are all equal in neutral stratification[6]. For the similarity functions in the range $-4 < z/L < -0.004$ of unstable stratification they recommend

$$\begin{aligned}
\phi_m &= \left(1 - 28\,\frac{z}{L}\right)^{-\frac{1}{4}} \\
\phi_h = \phi_w &= \left(1 - 14\frac{z}{L}\right)^{-\frac{1}{2}}
\end{aligned} \tag{8.32}$$

In stable stratification the result of Webb (1970) is usually used:

$$\phi_m = \phi_h = \phi_w = 1 + 5\frac{z}{L} \tag{8.33}$$

8.8.2 Profiles in stable stratification

If the ϕ functions for the stable side are written $\phi = 1 + \beta z/L$ (where $\beta \sim 5$) then the profiles are:

$$\begin{aligned}
\overline{u}(z) &= \frac{u_*}{k_v}\left(\ln\,\frac{z}{z_0} + \beta\,\frac{z - z_0}{L}\right) \\
\overline{T}(z) - \overline{T}(z_{0T}) &= \frac{T_*}{k_v}\left(\ln\,\frac{z}{z_{0T}} + \beta\,\frac{z - z_{0T}}{L}\right)
\end{aligned} \tag{8.34}$$

Since the diffusivities are all the same, all profiles have this log-linear form. In some contexts it is important to distinguish different roughness lengths for momentum and heat.

8.9 Velocity Standard Deviations

The "power in w," measured by ${\sigma_w}^2$, is controlled by a number of counterbalancing influences (notably buoyant production/extraction, redistribution to/from other components, vertical transport, and viscous dissipation). There is no reason to expect σ_w to have a simple distribution in the ASL. It must surely vanish "at" ground. But despite the complexity of the mechanisms which set the level of ${\sigma_w}^2$, to a rough first approximation the MOST prediction that

$$\frac{\sigma_w}{u_*} = \phi_{ww}\left(\frac{z}{L}\right) \tag{8.35}$$

[6] But see a later section that casts some doubt on the generality of this finding.

does apply, except in the UBL (unresolved basal layer) adjacent to ground (where $z \sim O[z_0]$), and which might be a tall plant canopy. There is modern evidence that $\phi_{ww}()$ has other arguments.

Formulae such as (Panofsky et al., 1977)

$$\frac{\sigma_w}{u_*} = \left[1.6 + 2.9(-z/L)^{2/3}\right]^{1/2} \tag{8.36}$$

or (Kaimal and Finngan, 1994)

$$\frac{\sigma_w}{u_*} = 1.25 \left(1 - 3\frac{z}{L}\right)^{1/3} \tag{8.37}$$

are clearly to be taken as (at best) giving σ_w/u_* *in the mean* for given z/L; but even at an ideal site, values over any observation interval may differ from these formulae by a large amount, eg. neutral values need not cluster very closely about $\sigma_w/u_* = 1.3$, at least on the evidence of this classic paper.

In the case of the horizontal components, it has been known for a long time that strict MO similarity does not apply. For example Calder (1966) noted that "the classical form of (MOST) cannot be applied legitimately to the variances of the horizontal components of the wind velocity fluctuation", and that this "has recently been suspected on the basis of observational data".

Townsend was the first to speak of "inactive turbulence", large, quasi-horizontal eddies with which is associated very little vertical motion and very little vertical transfer of momentum. The energy in the "inactive turbulence" is not correlated with the friction velocity, ie. the MOST velocity scale, and the lengthscale of those motions goes with BL depth δ. Consequently MOST is incomplete, as far as it pertains to $\sigma_{u,v}$ and it has been found to be useful to invoke δ as a further scale. According to Panofsky et al., and many other articles, the horizontal velocity standard deviations σ_u, σ_v in the unstably-stratified surface-layer obey the formula:

$$\frac{\sigma_{u,v}}{u_*} = [12 + 0.5(\delta/-L)]^{1/3} \tag{8.38}$$

Notice the implication for the 'turbulence intensity'. Taking the neutral case for simplicity, we have

$$\frac{u'}{\overline{u}} \sim \frac{\sigma_u}{\overline{u}} \approx \frac{2.3\ u_*}{(u_*/k_v)\ln(z/z_0)} \approx \frac{1}{\ln(z/z_0)} \tag{8.39}$$

which is small for large z/z_0. Thus in many (but not all) contexts we can neglect the fluctuation u' relative to the mean streamwise velocity. This implies the x axis is a '1-way' axis in many problems, in particular in (many) dispersion problems: throw a handful of dust in the air and it blows downwind.

8.9.1 Turbulent Prandtl and Schmidt Numbers

Ratios of the von Karman constants give us the turbulent Prandtl and Schmidt (S_c) numbers:

$$\begin{aligned} \frac{k_v}{k_{vh}} &= P_r = \left(\frac{K_m}{K_h}\right)_{|L|=\infty} \\ \frac{k_v}{k_{vc}} &= S_c = \left(\frac{K_m}{K_c}\right)_{|L|=\infty} \end{aligned} \tag{8.40}$$

Atmospheric "flux-gradient" experiments are not unanimous on the values of the von Karman constants for momentum, heat and vapour. An alternative and indirect source of information upon them stems from atmospheric tracer experiments. Eulerian dispersion models whose form reduces (in the limit of undisturbed, horizontally-uniform flow) to

$$\overline{u}\,\frac{\partial \overline{c}}{\partial x} = \frac{\partial}{\partial z}\left(K_c(z)\,\frac{\partial \overline{c}}{\partial z}\right) \tag{8.41}$$

with $K \propto {\sigma_w}^2$ apparently require $K_c/K_m \approx 1.6$ (ie. $S_c \approx 0.63$) for optimal agreement with the observations.

8.10 And finally, a quick indication of what's above the surface layer

The diurnal cycle in the components of the surface energy budget[7]

$$Q^* = Q_H + Q_E + Q_G \tag{8.42}$$

(where $Q_E = L\,E$ is the latent heat flux density corresponding to the evaporation rate E) provides much insight into the diurnal evolution of the (fair-weather, horizontally-uniform) ABL. For the time being, assume a dry system ($Q_E = 0$) and neglect the soil heat flux Q_G. Then the surface heat flux $Q_H \approx Q^*$, with Q^* large and positive by day (say, $Q^* \sim +500[W\ m^{-2}]$) due to strong net solar insolation, and moderately negative at night due to net longwave radiative loss ($Q^* \sim -100[W\ m^{-2}]$). Now, this strong surface heat flux implies daily dumping into the ABL of a substantial quantity of heat, with the converse at night, ie., a downward flow of heat to ground. Thus we expect instability (near the surface) by day and stability (temperature inversion) by night.

But recall that the buoyancy-production term $(g/T_0)\ \overline{w'T'}$ in the TKE equation enhances vertical exchange under unstable stratification (day), but damps vertical motion during stable stratification (by night). Hence we can expect the daytime ABL to be strongly mixed, and to mix the daily injection of heat though a deep layer (ABL depth δ or z_i growing large by day).

[7]We consider the vertical fluxes of energy through an imaginary reference plane just above the surface. A plane cannot store energy, so the net flux must vanish. Here Q^* is positive for a positive net flux of radiant energy directed downwards towards the ground-air interface; all other fluxes are defined as positive for energy flow away from the ground-air interface.

However the suppression of turbulence at night by a growing surface inversion usually means that heat gained by the ground from the atmosphere is extracted from a shallow turbulent layer (ABL depth δ shrinking by night).

A natural feedback, easily visualized as the TKE production term $(g/T_0)\ \overline{w'T'}$, prevents the development of extreme instability in the atmosphere, except immediately adjacent to ground, where the short time-scale of w' fluctuations limits mixing efficiency despite a strongly unstable temperature gradient. That is, away from the surface, buoyant production leading to strong mixing (large TKE) transports heat away from the surface layer into the bulk of the ABL along a temperature gradient that, due to the strength of mixing, need be only slightly on the unstable side of well-mixed (adiabatic). Ideally, then, the daytime CBL is "well-mixed", ie., its potential temperature $\overline{\theta} = \overline{\theta}(t)$ is height-independent (except in a shallow surface layer; see Fig.8.1) and increases steadily throughout the day due to the heat injection.

The above reasoning suggests the depth of mixing in the night-time ABL will be defined by the limited depth across which shear production is sufficiently large to overcome the buoyant suppression of turbulence, and so allow fluctuating vertical motion to be sustained.

Above the surface layer, the eddies are rather large (one might take δ as their length scale) and velocity statistics scale with the convective velocity scale

$$w_* = \left(\frac{g}{T_0}\ (\overline{w'T'})_{gnd}\ \delta \right)^{1/3} \tag{8.43}$$

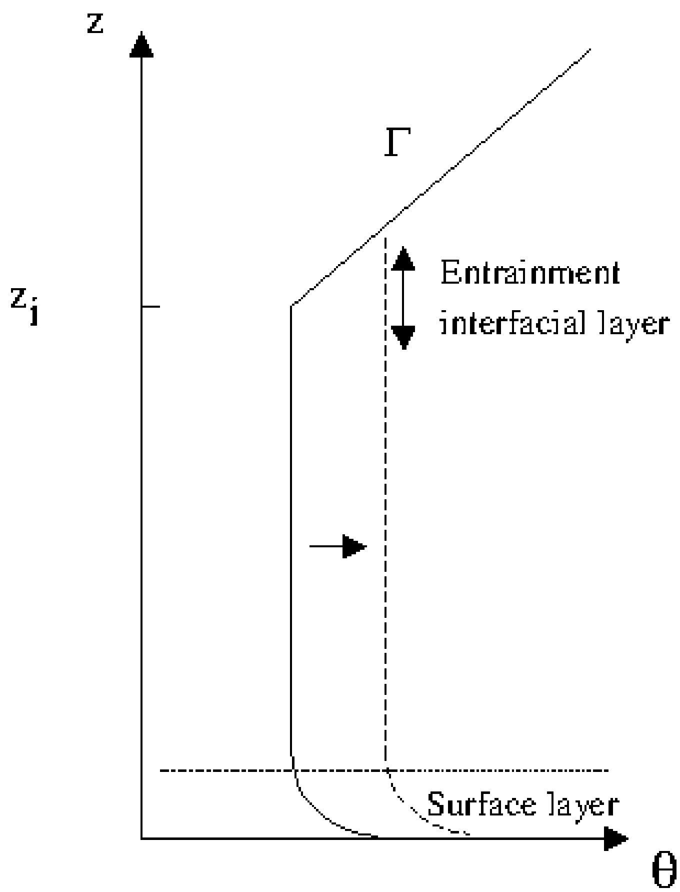

Figure 8.1: Idealized profile of mean potential temperature $\theta(z)$ in the daytime CBL (ie.unstable ABL). Highly-unstable surface layer, well-mixed (near-adiabatic) outer layer, capping inversion (lapse-rate of potential temperature Γ). Height of the base of the inversion z_i grows as θ increases due to surface heat input.

Bibliography

Anthes, R.A. 1977. A Cumulus Parameterization Scheme Utilizing a One-Dimensional Model. *Monthly Weather Review*, **105**, 270–286.

Calder, K.L. 1966. Concerning the Similarity Theory of A.S. Monin and A.M. Obukhov for the Turbulent Structure of the Thermally Stratified Surface Layer of the Atmosphere. *Quart. J. Roy. Meteorol. Soc.*, **92**, 141–146.

Dyer, A.J., & Bradley, E.F. 1982. An Alternative Analysis of Flux-Gradient Relationships at the 1976 ITCE. *Boundary-Layer Meteorol*, **22**, 3–19.

Garratt, J.R. 1992. *The Atmospheric Boundary Layer*. Cambridge University Press. 316pp.

Halstead, M.H., Richman, R.L., Covey, W., & Merryman, J.D. 1957. A Preliminary Report on the Design of a Computer for Micrometeorology. *J. Meteorology*, **14**, 308–325.

Kaimal, J.C., & Finnigan, J.J. 1994. *Atmospheric Boundary Layer Flows*. Oxford University Press. 289pp.

Panofsky, H.A., Tennekes, H., Lenschow, D.H., & Wyngaard, J.C. 1977. The Characteristics of Turbulent Velocity Components in the Surface Layer under Convective Conditions. *Boundary-Layer Meteorol*, **11**, 355–361.

Webb, E.K. 1970. Profile Relationships: The Log-Linear Range and Extension to Strong Stability. *Quart. J. Royal Meteorol. Soc.*, **96**, 67–90.

Yaglom, A. M. 1977. Comments on Wind and Temperature Flux-Profile Relationships. *Boundary-Layer Meteorol*, **11**, 89–102.