EAS 572: Assignment 1, Sept 2007 Value: 5%

Perform a dimensional analysis to determine the functional form of the TKE spectrum $S(f)$ in the inertial subrange.

Background

Suppose that $u'(t), v'(t), w'(t)$ are the time series of the velocity fluctuations over an interval of duration T (that is, u', v', w' are the instantaneous deviations from the respective averages $\overline{u}, \overline{v}, \overline{w}$ over the interval T). The 'turbulent kinetic energy' (TKE; common symbols k and E) of the turbulence (per unit mass) is defined

$$E \equiv \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) \equiv \frac{1}{2}\left(\sigma_u^2 + \sigma_v^2 + \sigma_w^2\right) \tag{1}$$

(in a more abstract way, we might say E is the TKE of the unresolved scales of motion). Note that $\sigma_u \equiv \sqrt{\overline{u'^2}}$ is the standard deviation of the velocity fluctuation u' over the interval.

Now, the velocity fluctuations have a random aspect, but also a certain persistence; this is a consequence of the fact that at any point and time, the local flow can be regarded as a superposition of many vortices of many sizes and lifetimes. Taking a statistical viewpoint, it is interesting and useful to characterize the distribution of 'eddy lifetimes', and this is given by the TKE spectrum, or 'power spectral density' $S(f)$ (where $0 \leq f \leq \infty$ is frequency) or $S(k)$ (where $0 \leq k \leq \infty$ is wavenumber). Taking the frequency representation, the power spectral density $S(f)$, shown schematically in Fig.1, has the property that

$$E = \int_0^\infty S(f)\, df \equiv \int_{-\infty}^\infty f\, S(f)\, d\ln f \ . \tag{2}$$

Now, viscosity must limit the upper end of the frequency spectrum, that is, velocity gradients cannot exist at infinitely small length scales, because viscosity would quickly smooth them out. Thus $S(f) \to 0$ as $f \to \infty$. Secondly, we expect (qualitatively) that the bulk of the TKE lies in or close to the 'production range' of eddy sizes, ie. scale which energy is fed into the turbulence: say f_p ('p' for 'peak' or 'production'). According to the paradigm of the "energy cascade," kinetic energy is passed 'downscale' from the production scale eddies to smaller and smaller eddies, an idea nicely expressed by L.F. Richardson's famous parody:

Big whorls have little whorls,

Which feed on their velocity;

And little whorls have lesser whorls,

And so on to viscosity.

The rate of TKE transfer down-scale (named ϵ, $\mathrm{m^2\,s^{-3}}$) is considered to be equal to the rate of conversion of TKE to heat. That is, according to this idealized spectral equilibrium of the mix of eddies, TKE is not "accumulated" in any scale-range but rather, is produced at rate ϵ at large scales, dissipated at rate ϵ at dissipation scales, and merely "passed through" the intermediate range of scales.

Kolmogorov gave a famous result for the shape of the power spectrum (usually cited in relation to the wavenumber spectrum; e.g. Stull, p390) in the 'inertial subrange,' i.e. frequency range $f_p \ll f \ll f_f \equiv 1/\tau_K$, where the Kolmogorov timescale identifies the 'fastest' eddies that viscosity will permit to exist in the flow; τ_K is uniquely determined by the two properties ϵ, ν. It is considered that the eddies responsible for the scales of motion constituting the inertial subrange have 'no memory' of the anisotropy of the production scale eddies, and that their kinetic energy is a function of the energy supply (ϵ) but is entirely unaffected by viscosity (ν), whose influence comes into play only on the much smaller/faster eddies of the dissipation range.

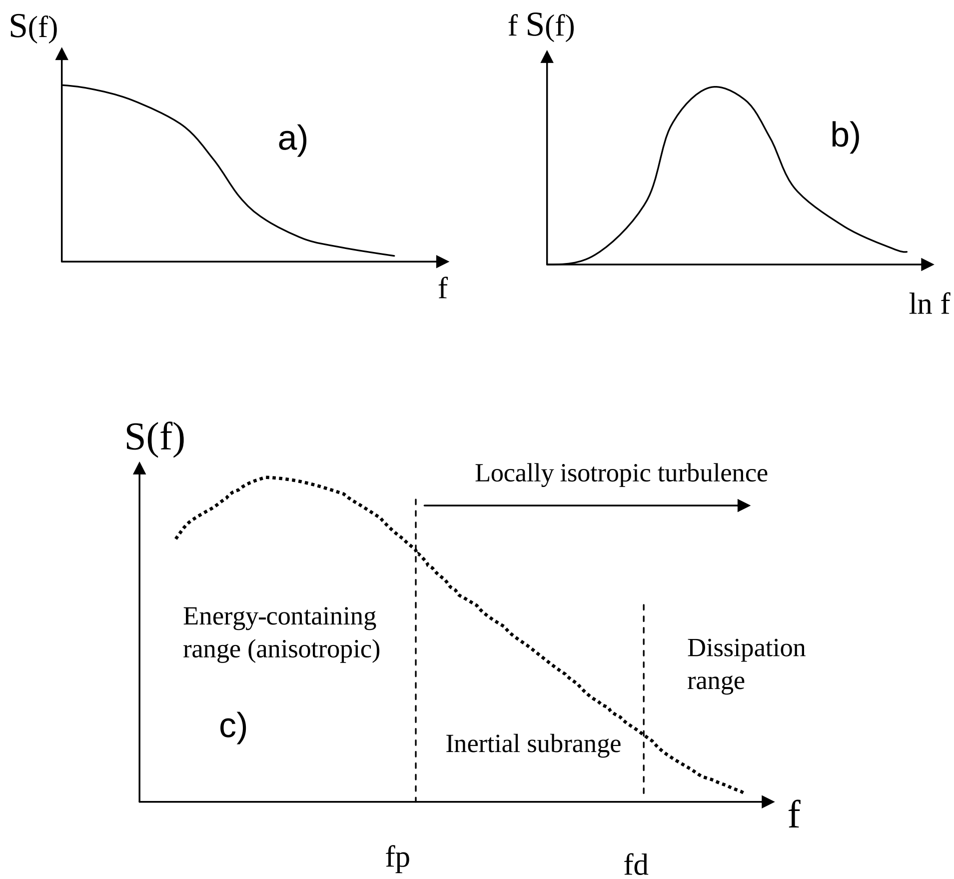

Figure 1: *The turbulent kinetic energy spectrum: area under the curve gives the turbulent kinetic energy E of the turbulence. Panel c identifies the inertial subrange.*