Representing Drag on Unresolved Terrain as a Distributed Momentum Sink

JOHN D. WILSON

Department of Earth and Atmospheric Sciences, University of Alberta, Edmonton, Alberta, Canada

(Manuscript received 5 April 2001, in final form 9 October 2001)

ABSTRACT

In numerical weather prediction models, drag on unresolved terrain is usually represented by augmenting the boundary drag on the model atmosphere, in terms of an effective surface roughness length. But as is shown here, if a terrain-following coordinate is defined relative to smoothed terrain, the residual unresolved terrain component implies a volumetric momentum sink, as recently implemented in the Canadian Climate Centre GCM, and as is implicit in the ''orographic-stress profile'' method. Thus treating drag on unresolved terrain by way of an internal (rather than enhanced surface) momentum sink is a better method in principle. While the skill of both methods hinges on limited fundamental knowledge of drag on terrain, a distributed momentum sink arguably offers greater flexibility to improve modeling of mountain winds, if necessary by tailoring the sink to achieve success, in specific regions, by trial and error.

A consequence of the new method is that unresolved terrain results in a ground-based (stress divergence) layer, that is somewhat analogous to a plant canopy layer, from the point of view of its momentum balance.

1. Introduction

Meteorologists in numerical weather prediction commonly consider that the improvement of model performance in mountainous regions is essentially an issue of grid resolution. But although improved spatial resolution has been shown to improve prediction of some variables, the field of vertical motion (and hence precipitation) remains problematical. This paper provides a simple argument for the representation of drag on unresolved terrain features by way of an *internal* (rather than enhanced *surface*) momentum sink, in order to improve modeling of mountain winds.

Smith (1978) and Davies and Phillips (1985) have measured terrain drag, by evaluating the height-integral

$$D/l = \int_{z_1}^{z_2} \Delta p(z)\, dz \qquad (1)$$

of the upwind–downwind pressure difference Δp across the terrain.[1] Davies and Phillips concluded that parameterization of terrain drag ''will need to be a suitable combination of an enhanced, non-trivial surface drag formulation to represent small scale and blocking effects, coupled with a free atmosphere momentum dissipation to represent the buoyancy wave contribution.''

Here we exclude from consideration wave drag and (Smith 1978) upstream blocking or regional trapping of cold surface air by topography. Some understanding of the drag of terrain on a *neutral* atmosphere has been derived from theoretical and numerical modeling studies,[2] which have progressed from consideration of low, smooth, isolated or periodic 2D terrain with low slope, to idealized 3D obstacles of large amplitude and slope. The mechanism of the drag of (periodic) terrain on a neutral atmosphere has been investigated in detail by Belcher et al. (1993), who identify and rank various contributions distinguishable from an asymptotic expansion in small parameters of the problem. For example they find that in the case of an undulating surface having low slope and a wavelength far less than the scale $U_0\, \delta/u_*$ (where δ is the depth of the boundary

[1] Here D/l is the drag per unit crosswind length, [N m^{-1}]; and if L is the alongwind length scale of the terrain, $D/(lL)$ gives the spatial mean drag [Pa]. Note that Eq. (1) derives from a surface integral on the terrain

$$D_i = -\int \sigma_{ij} \hat{n}_j\, dS \qquad (2)$$

for the net (vector) drag force, where σ_{ij} is the stress tensor, and $\hat{n}_j$ is the normal to the hill surface. Neglecting viscous stress and anticipating Reynolds averaging, $\sigma_{ij} = -\overline{p}\delta_{ij} - \rho\overline{u_i u_j}$. The pressure term dominates.

[2] These may be characterized by the linearized analytical solution of Jackson and Hunt (1975) for wind over low hills in the boundary layer, and subsequent developments of it (e.g., Hunt et al. 1988); by nonlinear numerical simulations using first-order (Taylor 1977) or second-order (Zeman and Jensen 1987) closure; and very recently by large eddy simulation (Brown et al. 2001). For reviews see Taylor et al. (1987), Finnigan (1988), and Taylor (1998).

Corresponding author address: Prof. John D. Wilson, Dept. of Earth and Atmospheric Sciences, University of Alberta, 1-26 Earth Sciences Building, Edmonton T6G 2E3, Canada.

layer, U_0 is a characteristic mean velocity, and u_{*0} is the unperturbed friction velocity), the dominant contribution to terrain drag is due to a surface pressure-asymmetry ''produced by the thickening of the perturbed boundary layer in the lee of the undulation'' (which mechanism they name ''nonseparated sheltering'').

It is clear from the theoretical studies that terrain drag depends on many parameters of the terrain and the atmosphere, but it is not the aim of this paper to formulate a more exact drag law for particular terrain. Rather, the subject here is how best to incorporate the drag of unresolved terrain (for argument's sake, assumed known, at least approximately), in atmospheric models.

Representation of drag on unresolved terrain

The drag of the ground on the atmosphere is commonly and conveniently quantified by the ''surface'' values (height $z \rightarrow 0$) of the turbulent shear stresses

$$\begin{aligned} \tau_{0x} &= -\rho \lim_{z\rightarrow 0} \overline{u'w'} \\ \tau_{0y} &= -\rho \lim_{z\rightarrow 0} \overline{v'w'}, \end{aligned} \tag{3}$$

where ρ is air density, (u', v', w') are the turbulent velocity fluctuations, and the overline (here) denotes a time average; the friction velocity u_{*0} is defined by $\rho^2 u_{*0}^4 = (\tau_{0x}^2 + \tau_{0y}^2)$ [see Weber (1999) for a discussion of nuances in the definition of u_*]. If one were to consider the microscopic details of the transference of this stress onto the solid earth, the drag could be decomposed into a viscous and a form-drag component. The drag τ_0 is often parameterized in terms of a geostrophic drag coefficient $C_g \equiv (u_{*0}/U_g)^2$ where U_g is the geostrophic wind speed, or, if referred to a local mean wind speed (S_p) at a height z_p within the atmospheric surface layer, may be parameterized in terms of a surface roughness length z_0, namely,

$$\sqrt{\frac{\tau_0}{\rho}} \equiv u_{*0} = \frac{k_v S_p}{\ln(z_p/z_0)}, \tag{4}$$

where $k_v \approx 0.4$ is von Kármán's constant (a diabatic correction is easily introduced). Note that according to Eq. (4), or rather the corresponding vector components of it along x, y, which involve component friction velocities u_{*0x}, u_{*0y}, the surface stress is aligned with the surface wind direction at height z_p.

In lieu of calculating height-varying terrain drag $\Delta p(z)$ on unknown or unresolved terrain, an effective net drag [as in Eq. (1)] may be applied at ''ground,'' as the surface stress τ_0. Fiedler and Panofsky (1972) introduced an ''effective'' roughness length, z_0^{eff}, as a ''space-average parameter,'' in order to estimate the correct momentum loss from the (model) atmosphere to ground, over uneven terrain.[3] This approach has been developed by others (e.g., Mason 1988; Xu and Taylor 1995) and is at present the usual way to treat drag on unresolved terrain. In numerical weather prediction (NWP) Eq. (4), with z_0 replaced by z_0^{eff}, allows one to calculate the local rate of momentum loss to ground, for a given value of the model wind speed (which, naturally, is to be regarded as a spatial average) at a near-ground grid point. Of course this approach is predicated on the supposition that the form of the mean wind profile over the terrain is (as for neutral flow on level terrain) semilogarithmic, a supposition supported by Hignett and Hopwood (1994), who find ''for heights proportional to only a small factor of terrain heights above the surface, the mean flow behaves logarithmically to a good approximation,'' and by Wood and Mason (1993), whose model results ''support the idea that sufficiently far above the hills, the areally-averaged velocity profile varies approximately logarithmically with height.'' Mason and King (1984), from their experimental study of flow over parallel ridges, noted that ''the influence of the terrain on the whole boundary layer is largely confined to a region close to the surface.''

As to specific knowledge of the augmented boundary drag, Wood and Mason (1993) proposed a drag formula integrating the known results, which, slightly rewritten following Xu and Taylor (1995), states:

$$\frac{D}{A_s \rho u_{*0}^2} = \alpha\beta\pi^2 \frac{\overline{u}^2(\mu z_m)}{\overline{u}^2(\mu h_m)} \frac{A_f^2}{A_s^2}. \tag{5}$$

In Eq. (5) A_s is the base-area (i.e., planform area) and A_f is the frontal area of the terrain feature; $A_s \rho u_{*0}^2$ is the residual drag force that would have occurred in the absence of topography; $\alpha = \alpha(\lambda/z_0)$ is an empirical dimensionless factor that absorbs a drag coefficient; β is a ''shape factor'' for the terrain ($\beta = 1$ for 2D terrain); and h_m, z_m are the height of the middle layer of the boundary-layer flow approaching the terrain (Hunt et al. 1988), and the pressure scale-height. This formula may not apply to wind over complex terrain, as opposed to a single, arbitrarily complex hill or ridge, because (Mason and King 1984) in general one may not regard the boundary layer approaching any one component of the topography as undisturbed; that is, there are limits to the validity of the linearized flow theory implicit in Eq. (5) by virtue of its reference to h_m, etc.

Scinocca and McFarlane (2000) reported having tested the representation of topographic wave drag in a GCM by means of the inclusion of a variable volumetric momentum sink in the momentum equations, at all model levels lying below the top of the (unresolved) terrain, the latter represented by means of elliptical barriers of appropriate height, eccentricity, and orientation. Similarly, as an alternative to the effective-roughness length method, Wood et al. (2001) tested the imposition of an ''explicit orographic-stress profile'' $\tau_{\text{oro}}(z) = \tau_{\text{oro}}(0)\, e^{-z/l}$, where $\tau_{\text{oro}}(0)$ was assumed to be given by Eq. (5) and where the decay length scale l was treated

[3] Alternatively, the influence of terrain may be parameterized by adjustment of the geostrophic drag coefficient, e.g., Sawyer (1959).

as being a constant; the divergence $\partial \tau_{\mathrm{oro}}/\partial z$ of the orographic stress is of course equivalent to a volumetric momentum sink (form drag). Scinocca and McFarlane noted that ''implementation of the form drag has the ability to change the direction of the flow so that it is more parallel to unresolved topographic ridges.'' While that effect can also be attained by the imposition of boundary (rather than internal) drag, as noted by Wood et al., the effective roughness length method necessarily aligns the surface shear stress with the low-level wind.

It is the purpose of this paper to give an explicit justification for the inclusion of orographic form drag in general, and to consider some of the consequences of that approach.

2. On the representation of terrain in atmospheric models

a. A smoothed map

The starting point for the representation of terrain is a digital map, say $h_{\mathrm{mp}}(x, y)$. Generally the available map is more intricate than could be represented by the model, which for argument's sake has grid spacings Δx, Δy that might be of order of 0.1–1 km (mesoscale model) or 10–100 km (weather model). Therefore we introduce smoothed terrain, that is, a map that has been smoothed on length scale δx, δy,

$$\tilde{h}(x, y) = \frac{1}{\delta x \delta y} \int_{-(1/2)\delta x}^{1/2\,\delta x} \int_{-(1/2)\delta y}^{(1/2)\delta y} G(x', y') \, h_{\mathrm{mp}}(x + x', y + y')\, dx'\, dy', \tag{6}$$

where $G(x', y')$ is the smoothing function.[4] Now we may decompose true terrain height $h(x, y)$ as

$$h(x, y) = \tilde{h}(x, y) + h'(x, y), \tag{7}$$

where $h'(x, y)$ is the local deviation of the elevation from the smoothed map, which, for the common smoothing functions, takes on both positive and negative values across the landscape. The field $h'(x, y)$ is sometimes accommodated statistically in weather models (''enhanced orography''). For example, using ''envelope mountains'' (Vernekar et al. 1992) one specifies the smoothed terrain height as

$$\tilde{h}_e(x, y) = \tilde{h}(x, y) + \alpha\, \sigma_h(x, y); \tag{8}$$

that is, one ''bumps up'' the model terrain by adding some multiple α of the standard deviation σ_h of h', over the grid cell. ''Silhouette mountains'' attempt to deal with less easily defined characteristics of the unresolved terrain, that is, its ''organization'' (random hills? or one or more linear ridges?), and its directional anisotropy (if ridgy, is there a predominating orientation?).

However, the smoothed map, $\tilde{h}$ or perhaps $\tilde{h}_e$, now becomes the hypothetical real lower surface of our atmosphere (see Fig. 1), and $\tilde{h}_x$, $\tilde{h}_y$ are the tangents of the slope angles of the smooth terrain (henceforth throughout the paper, subscripts x, y, z will indicate partial differentiation). At this point it is customary to worry no further about the topographic deviation h', other than to consider that mountains are to be characterized by an increased surface roughness length z_0^{eff} appearing in the treatment of the turbulent boundary layer by the ''physics package,'' and that unresolved gravity waves (caused by the unresolved terrain) may in some circumstances need to be parameterized.

[4] If G were a simple boxcar function, then δx, δy (alone) would define the range of terrain smoothing, and a choice $\delta x \sim \Delta x$ would retain about as much terrain resolution as the model grid permitted.

b. Choice of a vertical coordinate that is ''flat'' at the base of the model atmosphere

At this point we have the situation of Fig. 1. We wish to model the flow in the atmosphere above the curve $z = \tilde{h}(x, y)$, and we have the problem that we can not easily conform a grid to this surface, which to some extent retains the irregularity of the true terrain. We would like to impose natural boundary conditions on $z = \tilde{h}(x, y)$, but have the difficulty that what is ''natural'' is no longer clear, since $z = \tilde{h}(x, y)$ is not a true air/ground boundary.

To simplify grid generation, the strategy of most weather models is to choose a terrain-following vertical coordinate, that is, to perform a coordinate transformation into a new, nonorthogonal system of axes. The simplest example, and that used in this paper, is the distance η above local mean terrain height,

$$\eta(x, y, z) = z - \tilde{h}(x, y), \tag{9}$$

which satisfies $\eta = 0$ along the resolved terrain. Another common solution is a pressure-ratio coordinate (introduced by Philips 1957)

$$\sigma(x, y, z) = \frac{p(x, y, z, t)}{p(x, y, \tilde{h}(x, y), t)}, \tag{10}$$

which satisfies $\sigma = 1$ along the resolved terrain, and has the additional convenience of being ''flat'' at the top of the atmosphere [a property that also applies to a modified height coordinate, $\eta_* = (z - \tilde{h})/(z_T - \tilde{h})$, where z_T is the top of the model atmosphere]. As noted by Haltiner (1971), the use of pressure (or pressure ratio σ) as vertical coordinate is particularly attractive if it is to be assumed that the hydrostatic equation is valid.

In the following sections, the governing equations are expressed for the (x, y, η) coordinate system, but it will be obvious how to include in other terrain-following coordinate systems the new source terms due to interaction of the atmosphere with unresolved terrain. The important point is that, because of the coordinate transformation, there is no longer any terrain, in the sense of an irregular lower boundary to the atmosphere . . . rather, as shown below, the (resolved) terrain is ''felt'' by virtue of new source terms that arise in the (transformed) equations of motion.

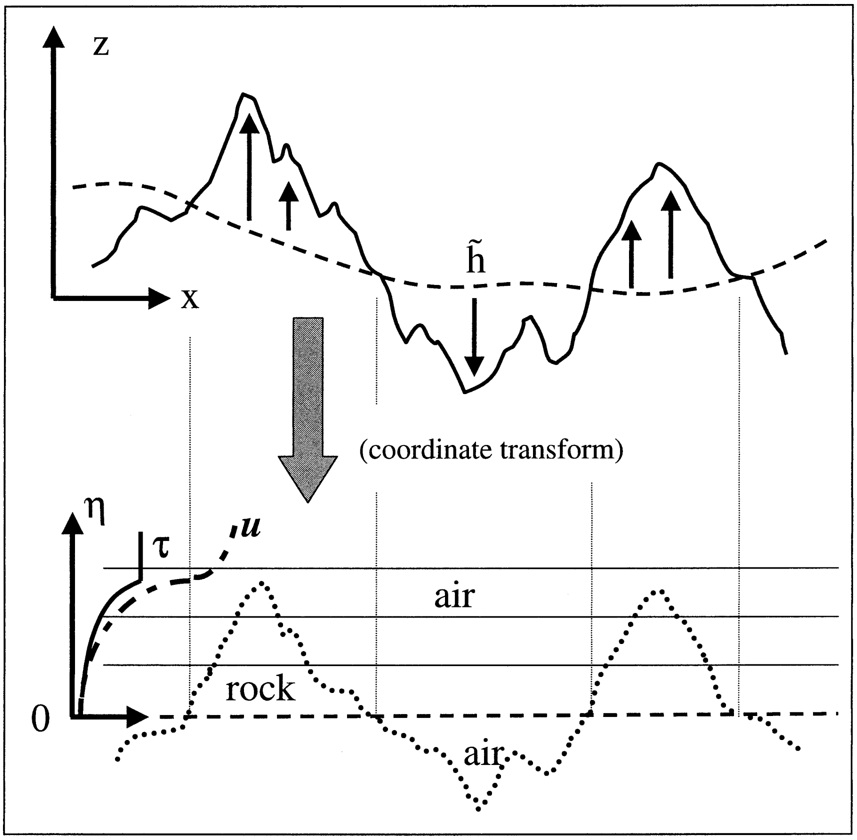

FIG. 1. Dashed-line in upper panel represents the smoothed terrain (height $z = \tilde{h}$), which upon coordinate transformation becomes the straight surface $\eta = 0$ in the lower panel; the vertical arrows represent the local (unresolved) terrain deviation $h'(x)$ from the smoothed terrain. The lower panel represents the configuration of a weather model in a terrain-following coordinate η, recognizing that the terrain ''followed'' leaves local topographic deviations unresolved.Schematic at lower left shows the stress divergence ($\partial\tau/\partial z$) across the layer of unresolved terrain, and resultant splitting of the (spatial-mean) wind profile (u) into an upper boundary-layer profile (concave upward) and a lower profile of opposite curvature in the terrain layer, joined at an inflexion point.

3. The conservation equations

Cauchy's equation of motion, valid for any continuum, states that [Batchelor 1985, Eq. (3.2.2)]

$$\frac{du_i}{dt} = F_i + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}, \qquad (11)$$

where u_i is the velocity of a fluid element, ρ is the density of the continuum, d/dt is the material (Lagrangian) derivative, F_i represents the sum of all body forces acting, and τ_{ij} is the stress tensor, whose divergence gives the sum of the surface forces upon the fluid element. The Navier–Stokes equations result when the stress tensor is expressed by way of Stokes's hypothesis for a Newtonian fluid, and under the Boussinesq approximation we may write the Navier–Stokes equations (conventionally) as

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + \nu\nabla^2 u_i + g\frac{\theta}{\theta_0}\delta_{i3} - 2\epsilon_{ijk}\boldsymbol{\Omega}_{ju_k}. \qquad (12)$$

Here p is the departure of the pressure from a hydrostatic and adiabatic reference state, ρ_0 is the mean density of the layer, θ is the departure of potential temperature from its reference value θ_0, $f = 2|\boldsymbol{\Omega}|$ are the components of the earth's angular velocity, and ν is the kinematic viscosity. We henceforth set $\rho_0 = 1$ so that p will represent the kinematic pressure departure, and neglect viscous momentum fluxes. We also simplify Coriolis terms by assuming the x axis parallel to lines of latitude, introducing the Coriolis parameter $f = 2|\Omega| \sin\phi$ (where ϕ is latitude), and neglecting small terms involving $\cos\phi$.

Bearing in mind that under the Boussinesq approximation the velocity field is nondivergent, an alternative expression of Eq. (12) is

$$\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_j}\left(u_i u_j + p\delta_{ij} - \nu\frac{\partial u_i}{\partial x_j}\right) + g\frac{\theta}{\theta_0}\delta_{i3} - 2\epsilon_{ijk}\mathbf{\Omega}_{j u_k}, \qquad (13)$$

where evidently $(u_i\, u_j + p\ \delta_{ij} - \nu(\partial u_i/\partial x_j))$ is a (kinematic) momentum flux. We have now a statement of (momentum) conservation in the universal form

$$\frac{\partial u_i}{\partial t} = -\mathbf{\nabla}\cdot\mathbf{F}_{u_i} + S_{u_i}, \qquad (14)$$

where the lhs is the ‘‘storage’’ term, and the rhs is the sum of the flux divergence (a transport term) and the source term(s).[5] As far as their effects on the local time evolution of u_i are concerned, a momentum-flux divergence and a source are equivalent.

4. Transforming the dynamical equations into the terrain-following coordinate

Once reframed in terms of the new, terrain-following coordinate, the model atmosphere is no longer disturbed by the complicated boundary condition of real terrain; that is, the no-slip and no-leak conditions applied relative to the resolved terrain $z = \tilde{h}(x, y)$. Rather, as this section shows, it will now be ‘‘driven’’ by source terms, or inhomogeneities, whose origin lies in the coordinate transformation from natural height above sea-level (z) to η (or σ or whatever other terrain-following coordinate system is chosen).

Momentum equations in the x, y, η coordinates

Let $F = F(x, y, z)$ be an arbitrary field, which we now rewrite as $F = F[x, y, \eta(x, y, z)]$. It is easy to show that

$$\left(\frac{\partial F}{\partial x}\right)_{yz} = \left(\frac{\partial F}{\partial x}\right)_{y\eta} + \left(\frac{\partial F}{\partial \eta}\right)_{xy}\left(\frac{\partial \eta}{\partial x}\right)_{yz}$$

$$\left(\frac{\partial F}{\partial y}\right)_{xz} = \left(\frac{\partial F}{\partial y}\right)_{x\eta} + \left(\frac{\partial F}{\partial \eta}\right)_{xy}\left(\frac{\partial \eta}{\partial y}\right)_{xz}$$

$$\left(\frac{\partial F}{\partial z}\right)_{xy} = \left(\frac{\partial F}{\partial \eta}\right)_{xy}\left(\frac{\partial \eta}{\partial z}\right)_{xy}, \qquad (15)$$

where in the third equation, for our particular coordinate η, $(\partial\eta/\partial z)_{xy} = 1$. We henceforth omit the clarifying subscripts on the partial derivatives.

It is straightforward using Eq. (15) to transform the partial derivatives appearing in the continuity equation and the Navier–Stokes equations. We retain horizontal (u, v) and vertical (w) velocities (i.e., velocities are still referred to fixed, local Cartesian axes), but it is advantageous to define

$$w^* \equiv w + \eta_x u + \eta_y v \equiv w - \tilde{h}_x u - \tilde{h}_y v \qquad (16)$$

as the difference between local vertical velocity and the projection of the horizontal velocity vector onto the local normal to the hill. In terms of w^*, the continuity equation may be written

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w^*}{\partial \eta} = 0, \qquad (17)$$

which amounts to a statement that the (air) mass flux-density vector

$$\mathbf{F}_a = \rho_0(u, v, w^*) \qquad (18)$$

is nondivergent in (x, y, η) space, and by implication, relative to the new coordinate system, convective fluxes are to be formed using velocity vector (u, v, w^*).

In the nonorthogonal, terrain-following frame of reference, the flux-form of the u momentum equation is

$$\frac{\partial}{\partial x}(u^2 + p) + \frac{\partial}{\partial y}(vu) + \frac{\partial}{\partial \eta}(w^*u) = fv + \frac{\partial}{\partial \eta}(\tilde{h}_x p). \qquad (19)$$

Note that in view of Eq. (17),

$$\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(vu) + \frac{\partial}{\partial \eta}(w^*u) \equiv u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w^*\frac{\partial u}{\partial \eta} \qquad (20)$$

so we may easily switch between the flux- and advection forms of the governing equations. The y- and w-momentum equations are

$$\frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(v^2 + p) + \frac{\partial}{\partial \eta}(w^*v) = -fu + \frac{\partial}{\partial \eta}(\tilde{h}_y p) \quad \text{and} \qquad (21)$$

$$\frac{\partial}{\partial x}(uw^*) + \frac{\partial}{\partial y}(vw^*) + \frac{\partial}{\partial \eta}(w^{*2} + p) = g\frac{\theta}{\theta_0} - \frac{\partial}{\partial x}(\tilde{h}_x u^2 + \tilde{h}_y uv) - \frac{\partial}{\partial y}(\tilde{h}_x uv + \tilde{h}_y v^2) - \frac{\partial}{\partial \eta}(\tilde{h}_x uw^* + \tilde{h}_y vw^*). \qquad (22)$$

The slope terms on the right-hand sides of the transformed momentum equations (19), (21), (22) ‘‘drive’’ the principal disturbance to the flow that we wish to result over the (smoothed) terrain, and have been written

[5] Generalizing from Eq. (14), there is a useful threefold classification of the terms arising in *any* of the conservation equations of fluid mechanics: storage terms are terms of form $\partial/\partial t(\ )$, i.e., partial derivatives in time; transport terms have form $\partial/\partial x_i(\ )$; and all other terms are sources.

as transport terms, though they may equally well be regarded as source terms, since $\partial/\partial\eta(\tilde{h}_x p) \equiv \tilde{h}_x \partial p/\partial\eta$ (etc).

5. The necessity to average the governing equations

When we use the Navier–Stokes equations, the issue of the meaning of the dependent variables always begs clarification. At the very least the equations are based on the ''continuum hypothesis'' and so ''u'' (etc.) is a volume average within a ''fluid element'' of a size very large with respect to the molecular mean free path, but small with respect to the smallest scales of convective motion, that is, small with respect to the Kolmogorov length.

But invariably in meteorology we wish the dependent variables to be representative over a finite (usually large) region. In fact, properties are usually interpreted as being volume and/or time averages (e.g., ''synoptic scale'' velocity field). When in recognition of that intended interpretation, proper averaging is applied to the conservation equations to obtain governing equations for the averages themselves, we have the Reynolds equations, which explicitly display the influence on the resolved flow $\overline{u}$ (etc.) of the unresolved flow, in the form of unresolved momentum fluxes.

To rationally represent terrain in an atmospheric model, there is need for even more care in defining the atmospheric variables, for consider the lower panel of Fig. 1. In some regions, unresolved positive terrain fluctuations ($h' > 0$) protrude through our lower coordinate surface $\eta = 0$ and into our atmosphere; elsewhere, $h' < 0$, and we have atmosphere below our nominal lower boundary. The consequence is that, whereas we should have liked to regard our ''flow variables'' as continuous functions of our model coordinates, $u = u(x, y, \eta)$ (etc.), the reality is that in some control volumes near the (resolved) terrain surface, certain positions $\mathbf{P} = (x, y, \eta)$ lie underground: at such locations, wind velocity is of course undefined. We do not have spatial continuity, because our atmosphere is ''multiply connected.''

This is a case important in the fluid mechanics of porous media and in agro-meteorology (Raupach and Shaw 1982; Finnigan 1985; Miguel et al. 2001) where the wind blows in and around vegetation. To define flow variables that are continuous functions of the coordinates in a multiply connected space, it is possible to introduce spatial averaging, and to derive the governing equations for the (continuous) averages.

A spatial average

Let $V = V_f + V_s = L_x L_y L_\eta$ be the volume over which we integrate to recover continuous variables, composed of a fluid subvolume (V_f) and a solid subvolume (V_s), and suppose we define our averaging process as

$$\overline{F}(x, y\ \eta) = \frac{1}{V_f}\int_{x-L_x/2}^{x+L_x/2}\int_{y-L_y/2}^{y+L_y/2}\int_{\eta-L_\eta/2}^{\eta+L_\eta/2} I(x', y', \eta') \times F(x', y', \eta')\, dx'\, dy'\, d\eta', \tag{23}$$

where the normalizing volume

$$V_f = \int_{x-L_x/2}^{x+L_x/2}\int_{y-L_y/2}^{y+L_y/2}\int_{\eta-L_\eta/2}^{\eta+L_\eta/2} I(x', y', \eta')\, dx'\, dy'\, d\eta'. \tag{24}$$

In Eqs. (23) and (24) $I(x', y', \eta')$ is an indicator function, having unit value if the point (x', y', η') lies within the atmosphere, and vanishing otherwise. In order to ensure that the indicator function does not vanish throughout an entire averaging volume, L_x, L_y need to be large with respect to the horizontal scales of the unresolved terrain field; but to retain sufficient vertical resolution of the vertical gradients presumably $L_\eta \ll L_x, L_y$ (averaging volumes are slabs). This choice for the averaging is termed by Miguel et al. (2001) the ''intrinsic (internal) phase average,'' while normalization using the total volume yields the ''superficial (external) phase average.'' The former seems natural in the present context, and in any case the two averages are trivially related by the porosity V_f/V.

Now we may perform a Reynolds decomposition $\mathbf{F} = \overline{\mathbf{F}} + \mathbf{F}'$, etc., and our resolved fields (velocity, etc.) are now (explicitly) defined as volume averages, and are spatially continuous. For ordinary spatial averaging in simply connected space, the operations of spatial differentiation and spatial integration commute; that is,

$$\overline{\frac{\partial \mathbf{F}}{\partial x_i}} \equiv \frac{\partial \overline{\mathbf{F}}}{\partial x_i} \tag{25}$$

and (for any vector $\mathbf{F}_i$)

$$\overline{\frac{\partial \mathbf{F}_i}{\partial x_i}} \equiv \frac{\partial \overline{\mathbf{F}}_i}{\partial x_i}. \tag{26}$$

In the multiply connected space (x, y, η), exchange of some arbitrary property (ϕ) of the atmosphere with the hidden terrain implies losses or gains of ϕ from the fluid subvolume of $L_x L_y L_\eta$; these exchanges are due to and representable as, surface integrals of exchange fluxes of ϕ across the air/solid boundaries. Bearing in mind the essential form of a fluid conservation equation,

$$\frac{\partial \phi}{\partial t} = -\nabla \cdot \mathbf{F}_\phi + S_\phi \tag{27}$$

the fluid–solid exchange of property ϕ (that we identify upon spatial averaging) evidently might be regarded as having to be accounted for, either by an (additional) apparent source, or by some modification of the flux-divergence term upon averaging. It is the latter interpretation that has been preferred, in the form of the ''spatial averaging theorem'' (e.g., Miguel et al. 2001;

Finnigan 1985) according to which Eq. (26) must be replaced by

$$\overline{\frac{\partial \mathbf{F}_i}{\partial x_i}} = \frac{\partial \overline{\mathbf{F}}_i}{\partial x_i} - \frac{1}{V}\int \mathbf{F}_i \hat{n}_i \, dS, \tag{28}$$

where the surface integral covers the *entire* surface area of unresolved terrain within $L_x L_y L_\eta$, and $\hat{n}_i$ is the local surface unit normal vector.

6. Continuum equations after spatial averaging

Since the fluxes of air across the surfaces of unresolved terrain vanish, no extra terms arise in the continuity equation upon spatial averaging, thus

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w^*}}{\partial \eta} = 0. \tag{29}$$

However on applying the spatial averaging to Eq. (19) for the u momentum, we obtain:

$$\begin{aligned} &\frac{\partial}{\partial x}(\overline{u}^2 + \overline{u'^2} + \overline{p}) + \frac{\partial}{\partial y}(\overline{u}\,\overline{v} + \overline{u'v'}) \\ &\quad + \frac{\partial}{\partial \eta}(\overline{u}\,\overline{w^*} + \overline{u'w^{*\prime}}) - f\overline{v} \\ &\quad = \tilde{h}_x \frac{\partial \overline{p}}{\partial \eta} - \frac{1}{V}\int [\tilde{h}_x p \hat{n}_\eta - (u^2 + p)\hat{n}_x \\ &\qquad - uv\hat{n}_y - uw^*\hat{n}_\eta] \, dS \end{aligned} \tag{30}$$

Had we retained viscous momentum fluxes in our original momentum equations, additional surface integrals of viscous stresses would have appeared. If h'_x, h'_y are the slopes of the unresolved terrain surface, then the projections of the normal vector onto the x, y, η axes are $\hat{n}_x = -h'_x/\sqrt{1 + h'^2_x + h'^2_y}$, $\hat{n}_y = -h'_y/\sqrt{1 + h'^2_x + h'^2_y}$, and $\hat{n}_\eta = 1/\sqrt{1 + h'^2_x + h'^2_y}$. Then Eq. (30) transforms to

$$\begin{aligned} &\frac{\partial}{\partial x}(\overline{u}^2 + \overline{u'^2} + \overline{p}) + \frac{\partial}{\partial y}(\overline{u}\,\overline{v} + \overline{u'v'}) \\ &\quad + \frac{\partial}{\partial \eta}(\overline{u}\,\overline{w^*} + \overline{u'w^{*\prime}}) - f\overline{v} \\ &\quad = \tilde{h}_x \frac{\partial \overline{p}}{\partial \eta} - \frac{1}{V}\int [(u^2 + p)h'_x + p\tilde{h}_x + uvh'_y \\ &\qquad - uw^*] \frac{dS}{\sqrt{1 + h'^2_x + h'^2_y}} \end{aligned} \tag{31}$$

(it is straightforward to obtain the corresponding equations for v, w^*).

On first sight it may appear that one could set all velocities to zero at the surface of the unresolved terrain, according to the no-slip, no-leak boundary condition. However recall we have not included the viscous terms, and nor will we ever in any direct way represent the microscopic topography, of clod and tussock. In the limit of small distance from the (true) terrain, a distance whose definition is necessarily vague, stresses are transferred onto ground by microscopic-scale form drag and by viscous drag, on the surface roughness elements. Ordinarily we represent this economically by relaxing the notion that convective stresses must vanish in the limit $z \to 0$, for example, on flat terrain we speak of the atmospheric surface layer as a "constant stress" layer, normally taken for convenience to imply that the turbulent convective shear stresses $\overline{u'w'}$, $\overline{v'w'}$ are constant, even as $z \to 0$.

From model studies of wind over hills, Wood and Mason (1993) found that "the perturbation to the net surface force is dominated by the pressure force," in agreement with earlier assessments. If we did assume the velocities vanish on the surface of the unresolved terrain,[6] Eq. (31) would simplify to

$$\begin{aligned} &\frac{\partial}{\partial x}(\overline{u}^2 + \overline{u'^2} + \overline{p}) + \frac{\partial}{\partial y}(\overline{u}\,\overline{v} + \overline{u'v'}) \\ &\quad + \frac{\partial}{\partial \eta}(\overline{u}\,\overline{w^*} + \overline{u'w^{*\prime}}) - f\overline{v} \\ &\quad = \tilde{h}_x \frac{\partial \overline{p}}{\partial \eta} - \frac{1}{V}\int p \frac{\tilde{h}_x + h'_x}{\sqrt{1 + h'^2_x + h'^2_y}} \, dS. \end{aligned} \tag{32}$$

In the last term on the rhs the slope $\tilde{h}_x$ of the smoothed terrain should typically be much smaller than the slope h'_x of the unresolved terrain.

7. Representation of the new terms

Apart from the distinction between w and w^*, a distinction that vanishes if the resolved slope is zero, the left-hand side of Eq. (32) is familiar, and must equate to zero if we had neither resolved, nor unresolved, terrain. In that case, we should have the conventional Reynolds equations. The Reynolds stresses, $\overline{u'w^{*\prime}}$, etc., express the influence on the resolved flow ($\overline{u}$) of the unresolved flow (i.e., the eddy motion hidden by the volume-averaging process).

In parameterizing the volumetric drag term, represented formally in Eq. (32) by the surface integral, it is a reasonable starting point to attempt to partition the bulk terrain drag (D), as represented, for example, by Eq. (5), into its distribution within the (unresolved) terrain layer, that is, to introduce a volumetric momentum sink:

$$C_{\mathrm{du}} a_u(x, y, \eta)\, \overline{u}\, \sqrt{\overline{u}^2 + \overline{v}^2} \tag{33}$$

with an equivalent term in the $\overline{v}$ equation. Here C_{du} is a drag coefficient, defined with respect to the local velocity; dimensionally a_u has units [1/length], and the normal interpretation of it would be that it represents a

[6] The question of whether or not we set velocities to zero on the unresolved surfaces is moot; these surface integrals will need to be parameterized anyway.

frontal area density [m²/m³]. The sink should have the property that, when integrated in a volume containing the unresolved terrain, the correct total drag results; that is,

$$\int_{-\Delta Y/2}^{\Delta Y/2} \int_{-\Delta X/2}^{\Delta X/2} \int_{\eta=0}^{\eta=\eta_{\max}} C_{\mathrm{du}}\, a_u(x, y, \eta) \times \overline{u}\, \sqrt{\overline{u}^2 + \overline{v}^2}\, dx\, dy\, d\eta = \frac{D}{\rho}. \quad (34)$$

The momentum lost from the airspace within any given control volume is transferred to the (unresolved) terrain, and thus constitutes a component of the total drag over any grid cell $\Delta x \Delta y$, and a torque on the earth.

The inclusion of these extra terms in the momentum equations will have the consequence that, except on absolutely level ground ($\tilde{h}$ locally constant), there can no longer be found a constant stress layer at the base of the (nominal) turbulent boundary layer [an atmospheric boundary layer is still presumed to exist, even over mountains, by the physics packages of present weather models]. This is analogous to the consequence of resolving a forest or plant canopy layer at the base of the atmospheric surface layer over level terrain, that is, once we take the step of explicitly representing ''internal'' drag, rather than simply applying an effective drag coefficient to obtain the momentum loss to the surface, a divergence of the vertical momentum flux will result, even in horizontally uniform flow.

Surface boundary condition

The unresolved terrain-height fluctuation h' about $\tilde{h}$ is as liable to be negative as positive. An immediate implication is that one is not compelled, in atmospheric models over smoothed terrain, to insist upon application of the no-slip (and no-leak) conditions along $z = \tilde{h}(x, y)$. Emeis (1987) has noted that for a steep valley ''a rotor forming in the valley prevents the main flow from entering the valley,'' while Mason and King (1984), from their study of flow over a succession of ridges, observe that ''when the flow is across the valley, the wind speeds in the valley are about 0.1 to 0.2 of those on the summit.'' Accordingly in some circumstances allowing free-slip, while inexact, may be a superior option to imposing no-slip, along $z - \tilde{h} = 0$. In fact, perhaps for the horizontal components one might write

$$\alpha \frac{\partial \overline{u}}{\partial \eta} + (1 - \alpha)\overline{u} = 0, \quad (35)$$

where $\alpha \leq 1$ provides a terrain-sensitive variation between no-slip ($\alpha = 0$) and free-slip ($\alpha = 1$). For example, perhaps

$$\alpha = \frac{\sigma_h \tilde{h}}{\Delta x \Delta y} \quad (36)$$

so that at sea level ($\tilde{h} = 0$), or over exceptionally even terrain ($\sigma_h = 0$), or wherever the area of the grid cell $\Delta x \Delta y$ is very large, we retain the usual treatment, that is, no-slip.

It is pertinent that in any region for which the new (internal) drag terms are significant in one or more layers near ground, the precise nature of the boundary condition applied (on $\overline{u}$, $\overline{v}$) is not very important: for, just as within a plant canopy, the distributed drag will ensure small velocities occur, in the limit as $z \to \tilde{h}$, irrespective of the condition that is imposed at the bottom boundary.

8. Conclusions

Regardless of the progress in grid refinement, over much of the earth's surface there will always remain an unresolved terrain component. Then if one wishes meteorological variables that are spatially continuous and have an exact interpretation, some complication ensues. If the dependent variables (velocity, temperature, etc.) are defined as spatial averages in order to ensure their spatial continuity,[7] then extra terms, in the form of surface integrals over unresolved terrain features, must appear in the governing equations of atmospheric (and also oceanic) models; while over a great part of the flow domain the extra terms may vanish *numerically*, they remain present *logically* in the momentum equations.

In effect, these terms imply only that the enhanced drag of formulas such as Eq. (5) is to be vertically distributed, which is exactly what is accomplished, albeit arbitrarily, by Wood et al. (2001) with their analytic profile ($e^{-z/l}$) of the explicit orographic stress. Modern NWP models provide many computational levels close to ground, and so are capable of rational inclusion of these terms; for example the Canadian Global Environmental Multiscale model has about seven model levels in approximately the lowest kilometer above ground. The distributed momentum sink will change the shape of the models' near-ground wind profiles, and reduce sensitivity of the near-ground wind to the (necessarily nonphysical) boundary condition imposed along $z = \tilde{h}(x, y)$.

What needs to follow is a parameterization of the distributed drag, accounting for the state of the (model) atmosphere (mean wind speed and direction; thermal stratification; and perhaps turbulent kinetic energy), and the type of unresolved terrain within any model grid cell $\Delta x \Delta y$. Particular cases would be isotropically distributed hills of circular base, or infinitely long ridges of a given orientation. The formulation must incorporate the degree of organization (vs randomness) in positioning of features, and interactions between terrain features (mutual sheltering). Such an approach provides an avenue to manipulate the vertical velocity field over spe-

[7] The approach of spatial averaging may not be the only solution; a reviewer wondered whether one might instead use line averages along streamlines.

cific regions, in an admittedly empirical but certainly rational way.

Acknowledgments. This work has been supported in part by a research grant from the Natural Sciences and Engineering Research Council of Canada.

REFERENCES

Batchelor, G. K., 1985: *An Introduction to Fluid Dynamics.* Cambridge University Press.

Belcher, S. E., T. M. J. Newley, and J. C. R. Hunt, 1993: The drag on an undulating surface induced by the flow of a turbulent boundary layer. *J. Fluid Mech.,* **249,** 557–596.

Brown, A. R., J. M. Hobson, and N. Wood, 2001: Large-eddy simulation of neutral turbulent flow over sinusoidal ridges. *Bound.-Layer Meteor.,* **98,** 411–441.

Davies, H. C., and P. D. Phillips, 1985: Mountain drag along the Gotthard section during ALPEX. *J. Atmos. Sci.,* **42,** 2093–2109.

Emeis, S., 1987: Pressure drag and effective roughness length with neutral stratification. *Bound.-Layer Meteor.,* **39,** 379–401.

Fiedler, F., and H. A. Panofsky, 1972: The geostrophic drag coefficient and the ''effective'' roughness length. *Quart. J. Roy. Meteor. Soc.,* **98,** 213–220.

Finnigan, J. J., 1985: Turbulent transport in flexible plant canopies. *The Forest–Atmosphere Interaction,* B. A. Hutchison and B. B. Hicks, Eds., D. Reidel, 443–480.

——, 1988: Air flow over complex terrain. *Flow and Transport in the Natural Environment: Advances and Applications,* W. L. Steffen and O. T. Denmead, Eds., Springer-Verlag, 183–229.

Haltiner, G. J., 1971: *Numerical Weather Prediction.* J. Wiley.

Hignett, P., and W. P. Hopwood, 1994: Estimates of effective surface roughness over complex terrain. *Boundary-Layer Meteor.,* **68,** 51–73.

Hunt, J. C. R., S. Leibovich, and K. J. Richards, 1988: Turbulent shear flows over low hills. *Quart. J. Roy. Meteor. Soc.,* **114,** 1435–1471.

Jackson, P. S., and J. C. R. Hunt, 1975: Turbulent wind flow over a low hill. *Quart. J. Roy. Meteor. Soc.,* **101,** 929–955.

Mason, P. J., 1988: The formation of areally-averaged roughness lengths. *Quart. J. Roy. Meteor. Soc.,* **114,** 399–420.

——, and J. C. King, 1984: Atmospheric flow over a succession of nearly two-dimensional ridges and valleys. *Quart. J. Roy. Meteor. Soc.,* **110,** 821–845.

Miguel, A. F., N. J. van de Braak, A. M. Silva, and G. P. A. Bot, 2001: Wind-induced airflow through permeable materials. Part 1: The motion equation. *J. Wind Eng. Indust. Aero.,* **89,** 45–57.

Philips, N. A., 1957: A coordinate system having some special advantages for numerical forecasting. *J. Meteor.,* **14,** 184–1851.

Raupach, M. R., and R. H. Shaw, 1982: Averaging procedures for flow within vegetation canopies. *Bound.-Layer Meteor.,* **22,** 79–90.

Sawyer, J. S., 1959: The introduction of the effects of topography into methods of numerical forecasting. *Quart. J. Roy. Meteor. Soc.,* **85,** 31–43.

Scinocca, J. F., and N. A. McFarlane, 2000: The parameterization of drag induced by stratified flow over anisotropic orography. Preprints, *Ninth Conf. Mountain Meteorology,* Boulder, CO, Amer. Meteor. Soc., 397–400.

Smith, R. B., 1978: A measurement of mountain drag. *J. Atmos. Sci.,* **35,** 1644–1654.

Taylor, P. A., 1977: Some numerical studies of surface boundary layer flow above gentle topography. *Bound.-Layer Meteor.,* **11,** 439–465.

——, 1998: Turbulent boundary-layer flow over low and moderate slope hills. *J. Wind Eng. Indust. Aero.,* **74–76,** 25–47.

——, P. J. Mason, and E. F. Bradley, 1987: Boundary-layer flow over low hills (a review). *Bound.-Layer Meteor.,* **39,** 107–132.

Vernekar, A., B. Kirtman, J. Zhou, and D. DeWitt, 1992: Orographic gravity-wave drag effects on medium-range forecasts with a general circulation model. *Physical Processes in Atmospheric Models,* D. R. Sikka and S. S. Singh, Eds., J. Wiley, 295–307.

Weber, R. O., 1999: Remarks on the definition and estimation of friction velocity. *Bound.-Layer Meteor.,* **93,** 197–209.

Wood, N., and P. Mason, 1993: The pressure force induced by neutral, turbulent flow over hills. *Quart. J. Roy. Meteor. Soc.,* **119,** 1233–1267.

——, A. R. Brown, and F. E. Hewer, 2001: Parameterizing the effects of orography on the boundary layer: An alternative to effective roughness lengths. *Quart. J. Roy. Meteor. Soc.,* **127,** 759–777.

Xu, D., and P. A. Taylor, 1995: Boundary-layer parametrization of drag over small-scale topography. *Quart. J. Roy. Meteor. Soc.,* **121,** 433–443.

Zeman, O., and N. O. Jensen, 1987: Modification of turbulence characteristics in flow over hills. *Quart. J. Roy. Meteor. Soc.,* **113,** 55–80.