Estimation of the Kolmogorov constant (C_0) for the Lagrangian structure function, using a second-order Lagrangian model of grid turbulence

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We review Sawford's [Phys. Fluids A **3**, 1577 (1991)] second-order Lagrangian stochastic model for particle trajectories in low Reynolds number turbulence, showing that it satisfies a well-mixed constraint for the (hypothetical) case of stationary, homogeneous, isotropic turbulence in which the joint probability density function for the fixed-point velocity and acceleration is Gaussian. We then extend the model to decaying homogeneous turbulence and, by optimizing model agreement with the measured spread of tracers in grid turbulence, estimate that Kolmogorov's universal constant (C_0) for the Lagrangian velocity structure function has the value of 3.0 ± 0.5. © *1995 American Institute of Physics.*

I. INTRODUCTION

In this paper we are concerned with the numerical value of the universal (?) constant C_0 that appears in Kolmogorov's theoretical small-time estimate

$$D_{ij}(\Delta t)=C_0\epsilon\delta_{ij}\Delta t \tag{1}$$

for the Lagrangian velocity structure function

$$D_{ij}(\Delta t)=\langle[U_i^+(t+\Delta t)-U_i^+(t)][U_j^+(t+\Delta t)-U_j^+(t)]\rangle. \tag{2}$$

Here the bracket $\langle\,\rangle$ denotes the expected value of its contents; U_i^+ is the Lagrangian velocity. [We use $\mathbf{U}^+,\mathbf{A}^+=\partial\mathbf{U}^+/\partial t$ for Lagrangian velocity and acceleration; $\mathbf{U},\mathbf{A}=d\mathbf{U}/dt$ denote the fixed point (Eulerian) velocity and acceleration fields.] t and $t+\Delta t$ are arbitrarily separated times; ϵ is the mean rate of dissipation of turbulent kinetic energy; and Δt in Eq. (1) is a time increment satisfying $t_\eta\ll\Delta t\ll T_L$, where t_η is the Kolmogorov inner time scale, and T_L is the integral time scale. Our interest in C_0 stems from the fact that predictions of turbulent dispersion, if obtained using Lagrangian stochastic (LS) models satisfying the criteria provided by Thomson,[1] which include consistency with Eq. (1), will depend upon the value taken for it. That this is so is seen most easily in the case of homogeneous, stationary turbulence; for C_0 is then[2] related to the Lagrangian timescale by

$$T_L=\frac{2\sigma_V^2}{C_0\epsilon}. \tag{3}$$

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In principle the "true" value of C_0 could be determined from investigations of any turbulent flow, and widely differing means to do so have been exercised i.e., Lagrangian velocity measurements; direct numerical simulations (DNS); the observed dispersion of tracer particles in a flow. It is perhaps not surprising that a wide range of estimates of C_0 is to be found in the literature.[3] Luhar and Britter[4] and Du *et al.*[5] obtained (qualitatively) adequate predictions of dispersion from sources in the convective boundary layer (CBL), using well-mixed[1] LS models with $C_0=2.0$. Wilson *et al.*[6] compared predictions of a well-mixed LS model with the (numerous and definitive) Project Prairie Grass observations[7] of atmospheric surface layer dispersion, and obtained excellent quantitative agreement with (in effect) the specification $C_0=3.1$. Hanna[8] suggested $C_0=4.0\pm2.0$, on the basis of Lagrangian velocity measurements (neutrally-buoyant balloons) in the CBL. Sawford[9] suggested $C_0=7.0$, by comparing the ratio T_L/t_η as obtained from a second-order Lagrangian stochastic model with the value calculated from Yeung and Pope's[10] DNS of homogeneous isotropic turbulence. And at the upper end of the range suggested, Sawford and Guest[11] found $5\leqslant C_0\leqslant10$ yielded best simulations of dispersion within a physically-modeled neutral boundary layer.

When, as has sometimes been the case, C_0 is inferred from measured tracer dispersion, the value obtained depends on the correctness (or otherwise) of the dispersion model. First order LS models presume the joint evolution of position and velocity (X_i^+,U_i^+) to be Markovian. This is defensible for large Reynolds number turbulence, but at low Reynolds number a better assumption is that position, velocity and acceleration are jointly Markovian (second-order LS model). Sawford[9] suggested that variations in the Reynolds number, across the various dispersion experiments available, may account for the variability in estimates of C_0 obtained using first-order LS models. By introducing a second-order LS

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model (in which the Reynolds number is explicitly incorporated and C_0 is *truly* independent of it) for homogeneous, stationary, and isotropic turbulence, he showed that in first-order models the *supposedly* universal constant C_0 is *not* universal, but rather depends on the Reynolds number—Reynolds number effects are manifested in first-order models through nonuniversality across different flows of the "best" value of C_0.

Our objective here then, is to use a Lagrangian stochastic dispersion model (of known pedigree) to infer from measurements of dispersion in the very simplest of turbulent flows, the true value of C_0. To this end, we will first review the physical basis of Sawford's model. By broadening the well-mixed constraint to encompass acceleration, we will show that the Sawford model is uniquely correct for homogenous, stationary, isotropic turbulence; only provided it is a satisfactory assumption that for such turbulence the joint probability density function (PDF) for the Eulerian velocity and acceleration is Gaussian (Gaussianity of that PDF was not explicitly assumed by Sawford). Then by extending the model to decaying turbulence, we will evaluate the optimal value of C_0, by fitting model predictions to laboratory measurements of tracer spread in grid turbulence.

II. SAWFORD'S SECOND-ORDER MODEL

Consider isotropic, homogeneous, and stationary turbulence, and let (Z^+,W^+,A^+) be one component of the position, velocity, and acceleration of a tracer particle. Assuming that the collective evolution of (Z^+,W^+,A^+) is Markovian (for very high Reynolds number, it is usually assumed that the evolution of velocity and position is jointly Markovian), one has the (otherwise general) model:

$$\begin{aligned} dA^+&=a(A^+,W^+,Z^+,t)dt+b(A^+,W^+,Z^+,t)d\zeta(t),\\ dW^+&=A^+dt, \qquad\qquad (4)\\ dZ^+&=W^+dt, \end{aligned}$$

where $\zeta(t)$ is a Wiener process. Sawford[9] assumed within this overall framework a particular form for $a(Z^+,W^+,A^+,t)$, namely

$$a=-\alpha_1 A^+-\alpha_2 W^+. \qquad (5)$$

It can be shown that choice (5) *implies* (by virtue of Thomson's[1] well-mixed condition) a joint Gaussian PDF for A,W. However for our purpose it helps to turn the argument around: we will presently *assume* the Eulerian (A,W) statistics Gaussian, and *deduce* the form of $a(Z^+,W^+,A^+,t)$.

The stochastic differential equations (4) imply a governing equation, the Fokker–Planck equation, for the evolution of the joint PDF $p(Z^+,W^+,A^+,t)$:

$$\frac{\partial p}{\partial t}=-\frac{\partial}{\partial Z^+}(W^+p)-\frac{\partial}{\partial W^+}(A^+p)-\frac{\partial}{\partial A^+}(ap)+\frac{1}{2}\frac{\partial^2}{\partial A^{+2}}(b^2p). \qquad (6)$$

Now, we extend Thomson's well-mixed constraint by the following proposition:If at time $t=t_0$, p is proportional to p_a, the Eulerian joint PDF of the acceleration, velocity and position, then at a later time $t>t_0$, p must remain to be proportional to p_a. Mathematically this requires that p_a be a solution of Eq. (6). So, we have

$$\frac{\partial p_a}{\partial t}=-\frac{\partial}{\partial Z}(Wp_a)-\frac{\partial}{\partial W}(Ap_a)-\frac{\partial}{\partial A}(ap_a)+\frac{1}{2}\frac{\partial^2}{\partial A^2}(b^2p_a). \qquad (7)$$

In homogeneous and stationary turbulence this requirement reduces to

$$-\frac{\partial}{\partial W}(Ap_a)-\frac{\partial}{\partial A}(ap_a)+\frac{1}{2}\frac{\partial^2}{\partial A^2}(b^2p_a)=0. \qquad (8)$$

We now introduce the assumptions upon which, in effect, the Sawford model rests. First, we assume the Eulerian velocity PDF to be Gaussian; this is supported by experimental data from homogeneous and isotropic turbulence.[12] Second, we assume that the Eulerian acceleration PDF is also Gaussian (the validity of this assumption is explored in the Appendix). In stationary turbulence, velocity and acceleration are uncorrelated, and so in this case we obtain for the Eulerian joint PDF of velocity and acceleration:

$$p_a=\frac{1}{2\pi\sigma_W\sigma_A}\exp\left(-\frac{W^2}{2\sigma_W^2}-\frac{A^2}{2\sigma_A^2}\right), \qquad (9)$$

where σ_W and σ_A are the standard deviations of the velocity and the acceleration, respectively. Substituting into Eq. (8), we obtain

$$a=-\frac{b^2}{2\sigma_A^2}A^+-\frac{\sigma_A^2}{\sigma_W^2}W^+. \qquad (10)$$

By requiring his model to yield an asymptotically stationary random process $A^+(t)$, Sawford from his assumption [Eq. (5) here] found

$$b=\sqrt{2\alpha_1\alpha_2\sigma_W^2}, \qquad (11)$$

where in view of our Eq. (10),

$$\alpha_1=\frac{b^2}{2\sigma_A^2},\quad \alpha_2=\frac{\sigma_A^2}{\sigma_W^2}. \qquad (12)$$

It is obvious that this stationarity property is satisfied by the present (more general) model. This is not surprising because Thomson's well-mixed constraint encompasses the condition of the asymptotic stationarity of a random process.[1]

Equation (10) automatically gives the correct velocity structure function in the dissipation range. It is desirable that it also yields the correct correlation function in the inertial subrange. Following Sawford, this is ensured if we specify

$$b=\sqrt{\frac{2\sigma_W^2}{T^3}\,\mathrm{Re}_*(1+\mathrm{Re}_*^{-1/2})}, \qquad (13)$$

where

$$\mathrm{Re}_* = \frac{16 a_0^2}{C_0^4}\,\mathrm{Re}, \quad T = \frac{2\sigma_W^2}{C_0 \epsilon},$$
$$\mathrm{Re} = \left(\frac{T_E}{t_\eta}\right)^2, \quad T_E = \frac{\sigma_W^2}{\epsilon}, \tag{14}$$

T_E is an Eulerian time scale and T is a Lagrangian time scale. The dimensionless constant a_0 is defined by[13]

$$\sigma_A^2 = \frac{a_0 \epsilon}{t_\eta}. \tag{15}$$

This is obtained by dimensional analysis in the framework of Kolmogorov's second hypothesis: that in locally homogeneous and isotropic turbulence, the motion is determined by the forces of viscous friction and inertia.[14] For very high Reynolds number, a_0 is universal; but when the Reynolds number is finite, a_0 can be Reynolds-number dependent.[10]

Since b, if specified by Eq. (13), is independent of both W^+ and A^+, then a in Eq. (10) will be linear in W^+ and A^+. This is the property assumed by Sawford as a precondition of his model for homogeneous, stationary, and isotropic turbulence. It follows from our reexamination of that model that since the PDF of acceleration cannot be exactly Gaussian (see Appendix), the Sawford model cannot be exactly correct.

III. EXTENSION OF THE SAWFORD MODEL TO DECAYING TURBULENCE

In any real flow, energy dissipation ensures that the turbulence cannot be both stationary and homogeneous. In this section we extend the Sawford model to homogeneous decaying turbulence, in order to develop a model applicable to decaying grid turbulence.

In nonstationary turbulence, the Eulerian velocity and acceleration are correlated:

$$\rho = \frac{\langle WA \rangle}{\sigma_W \sigma_A} = \frac{1}{\sigma_W \sigma_A}\left\langle W \frac{dW}{dt}\right\rangle = \frac{1}{2\sigma_W \sigma_A}\frac{d}{dt}\langle W^2 \rangle \neq 0. \tag{16}$$

Continuing to assume the Eulerian PDF for velocity and acceleration is a joint Gaussian, we then have

$$p_a = \frac{1}{2\pi \sigma_W \sigma_A \sqrt{1-\rho^2}} \times \exp\left[-\frac{\sigma_W^2 A^2 + \sigma_A^2 W^2 - 2\rho \sigma_W \sigma_A W A}{2\sigma_W^2 \sigma_A^2 (1-\rho^2)}\right]. \tag{17}$$

Equation (7) yields

$$a = -\left[\frac{b^2}{2\sigma_A^2(1-\rho^2)} - \rho\frac{\sigma_A}{\sigma_W} - \frac{\sigma_A'}{\sigma_A} + \frac{\rho\rho'}{1-\rho^2}\right]A^+ - \left[-\frac{b^2\rho}{2\sigma_A\sigma_W(1-\rho^2)} + \frac{\sigma_A^2}{\sigma_W^2} - \frac{\rho'}{1-\rho^2}\frac{\sigma_A}{\sigma_W}\right]W^+. \tag{18}$$

The symbol ($'$) represents the derivative with respect to time. It is interesting that in this slightly more complicated turbulence the second-order model remains linear in A^+ and W^+. Equation (18) reduces to the original Sawford model [Eq. 10] for stationary turbulence. Since the statistics of the increment of acceleration A^+ are mainly determined by small scale eddies, under the hypothesis of local isotropy b remains, as given by Eq. (13), even in decaying turbulence.

IV. THE MAGNITUDE OF C_0

In second-order trajectory models, the constant C_0 is free of the Reynolds-number effects and is therefore genuinely universal. This property makes it possible to determine C_0 by fitting-second-order model predictions to experimental data. In grid turbulence the collective assumptions of homogeneity, isotropy, and Gaussianity (of the velocity PDF) are approximately satisfied. Therefore, we will use Eqs. (4), (13), and (18) to predict turbulent dispersion in water channel and wind tunnel grid turbulence.

A. Simulation of water channel dispersion

Measurements of the dispersion of a neutrally-buoyant saline tracer released from a point source into decaying homogeneous turbulence (grid turbulence) have been carried out in a water channel at the University of Alberta. A detailed description of the experiment has been given by Wilson *et al.*,[15] and here we list only the turbulence statistics needed in order to simulate tracer trajectories using the present model:

$$\sigma_W = 0.13\,U\left(\frac{X+X_0}{M} - 13\right)^{-1/2},$$
$$\sigma_U = 0.195\,U\left(\frac{X+X_0}{M} - 6.5\right)^{-1/2}. \tag{19}$$

Here $U = 18.75\ \mathrm{cm\,s^{-1}}$ is the mean alongstream velocity; $M = 7.62$ cm is the center-to-center mesh spacing; X is the downstream distance from the source to the point of interest; and X_0 is the distance from the grid to the source ($X_0 = 147.5$ cm).

In decaying homogeneous turbulence the turbulent kinetic energy budget is approximately a balance between the dissipation rate ϵ and advection by the mean flow, i.e.,[16]

$$\epsilon \approx -U\frac{\partial k}{\partial X} = -\frac{U}{2}\frac{\partial}{\partial X}(\sigma_U^2 + \sigma_V^2 + \sigma_W^2). \tag{20}$$

Because v was not measured, we assume that $\sigma_v = \sigma_w$.

To specify a_0 we used Pope's[17] formula

$$a_0 = 3\left(1 - \frac{22}{\mathrm{Re}_\lambda}\right), \tag{21}$$

where $\mathrm{Re}_\lambda = \sigma_w \lambda/\nu$ is the Reynolds number based on the Taylor microscale $\lambda = (15\nu\sigma_w^2/\epsilon)^{1/2}$. This formula is derived from the DNS data of Yeung and Pope.[10]

Figure 1 compares the measured and predicted standard deviation of vertical spread σ_z, for several assumed values of C_0. The choice $C_0 = 3.0 \pm 0.5$ gives a good fit of the second-order model to the experimental data.

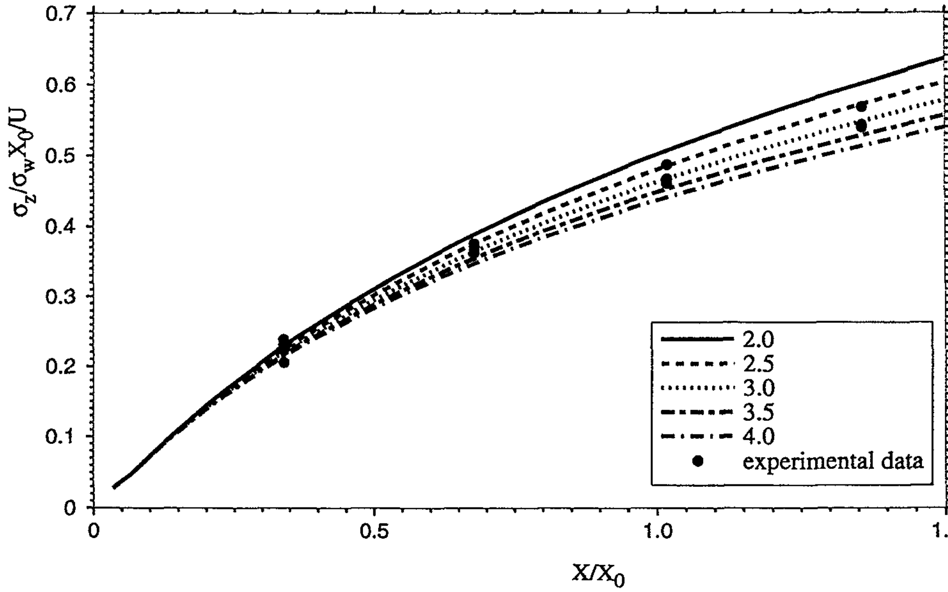

FIG. 1. Vertical dispersion from a tracer source in a water channel experiment, compared with simulations using a second-order LS model with different values of C_0.

B. Simulation of wind tunnel dispersion

The rate of dispersion was measured in decaying grid turbulence in a wind tunnel at Division of Atmospheric Research, CSIRO, Australia. Best-fit formulae for turbulence velocity statistics are

$$\sigma_U = 0.060U\left(\frac{X+X_0}{X_0}\right)^{-0.74},$$

$$\sigma_V = 0.055U\left(\frac{X+X_0}{X_0}\right)^{-0.71}, \quad (22)$$

$$\sigma_W = 0.053U\left(\frac{X+X_0}{X_0}\right)^{-0.69},$$

where $U(=548\ \mathrm{cm\ s^{-1}})$ is the mean velocity along the wind tunnel, X is the streamwise distance from the source, and $X_0=31.0$ cm is the distance from the grid to the source. We estimated dissipation rate ϵ by the means indicated earlier.

Figure 2 compares measured and calculated vertical spread of the tracer. As in the case of the water channel data, $C_0=3.0\pm0.5$ gives a good fit.

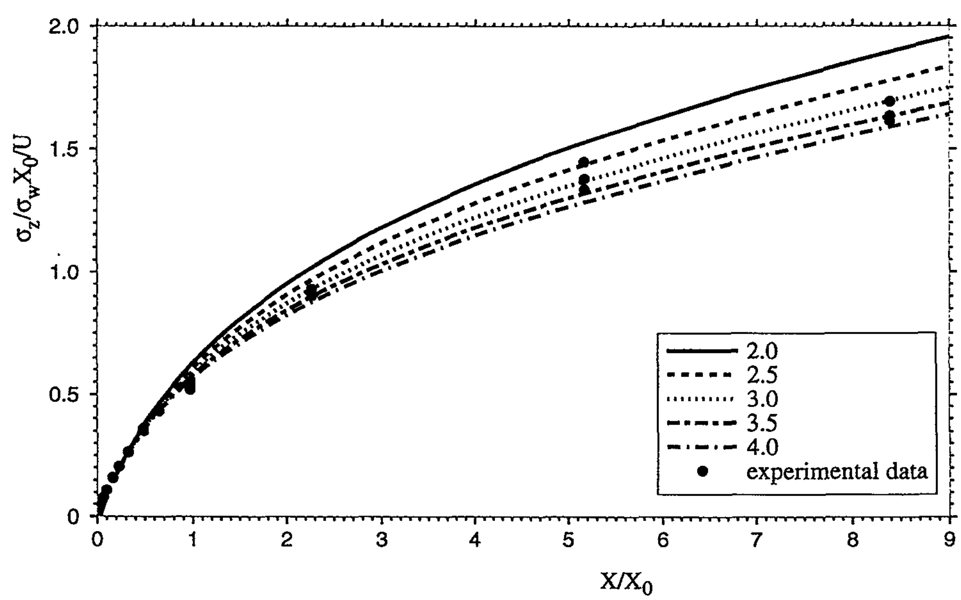

FIG. 2. Vertical dispersion from a tracer source in a wind tunnel experiment, compared with simulations using a second-order LS model with different values of C_0.

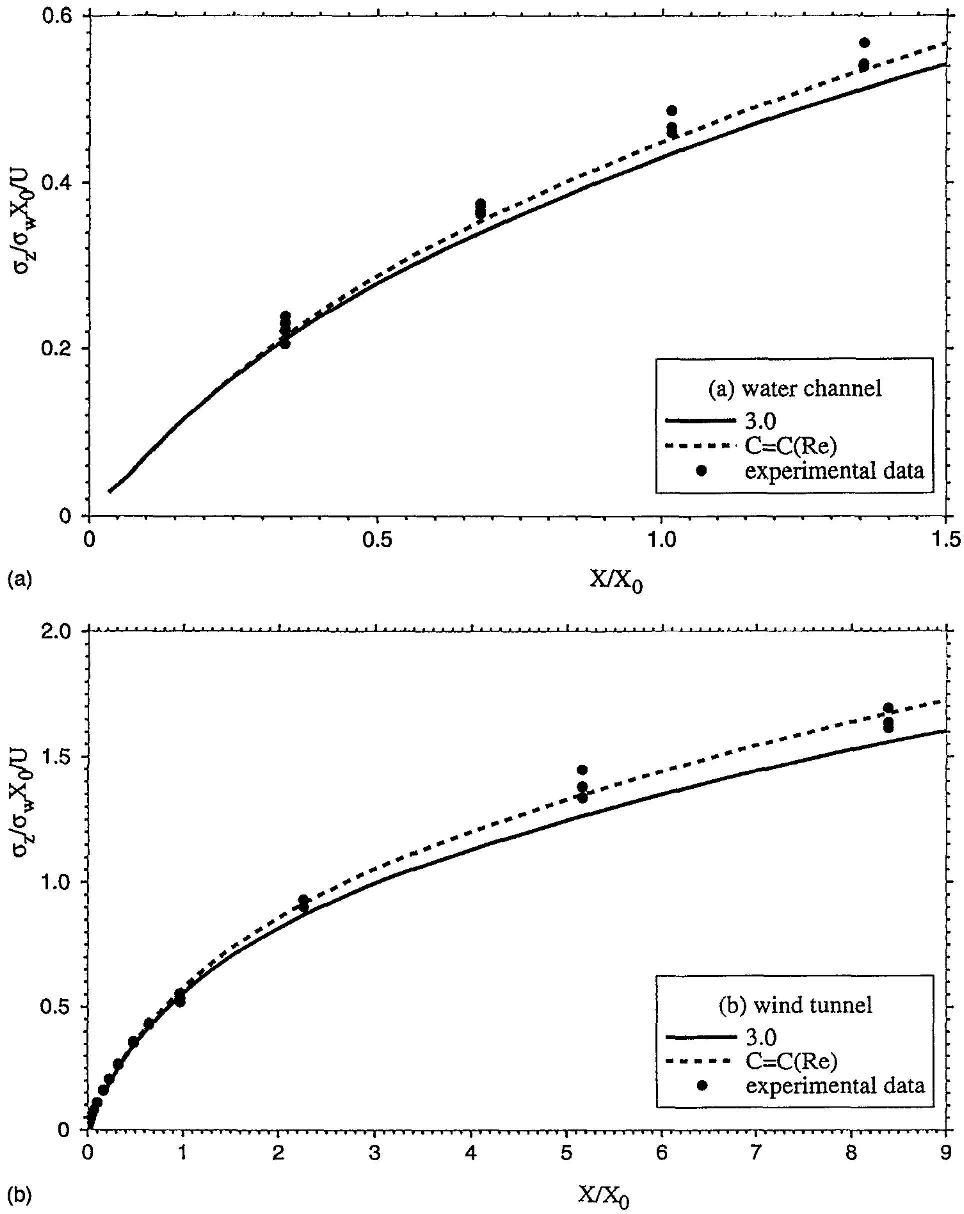

FIG. 3. Vertical dispersion from a tracer source compared with simulations using a first-order model (with and without Reynolds number correction): (a) water channel, (b) wind tunnel.

C. Estimates of C_0 from infinite Reynolds number flow

The Reynolds number for atmospheric boundary layer turbulence is (effectively) infinite. Rodean[3] estimated that in the neutral atmospheric surface layer (NSL), the Kolmogorov constant $C_0 \approx 5.7$. The basis for this result (or its equivalent for the specification of a Lagrangian time scale) is as follows.

Suppose in the NSL we regard the Eulerian velocity statistics as Gaussian (this is quite a good assumption, except within or close to the vegetation, i.e., provided height $Z \gg z_0$, where z_0 is the surface roughness length). Thomson[1] proved that the model

$$dW^+ = -\frac{W^+}{T(Z)}\,dt + \frac{1}{2}\,\frac{\partial \sigma_W^2}{\partial Z}\left(1 + \frac{W^{+2}}{\sigma_W^2}\right)dt + b\,d\zeta, \quad (23)$$

where

$$b = \sqrt{C_0 \epsilon}, \quad T(Z) = \frac{2\sigma_W^2}{C_0 \epsilon} \quad (24)$$

is the uniquely correct one-dimensional model for Gaussian inhomogeneous turbulence; that is, it is the "uniquely correct" model within the most rigorous theoretical framework presently available, that of Thomson.[1] This model is easily shown to be equivalent to

$$d\left(\frac{W^+}{\sigma_W}\right) = -\frac{W^+}{\sigma_W}\,\frac{dt}{T} + \frac{\partial \sigma_W}{\partial Z}\,dt + \sqrt{\frac{2}{T}}\,d\zeta \quad (25)$$

which is the infinitesimal form of the model compared by Wilson *et al.*[6] (hereafter WTK) against the Project Prairie Grass field observations of dispersion. [The equivalence be-

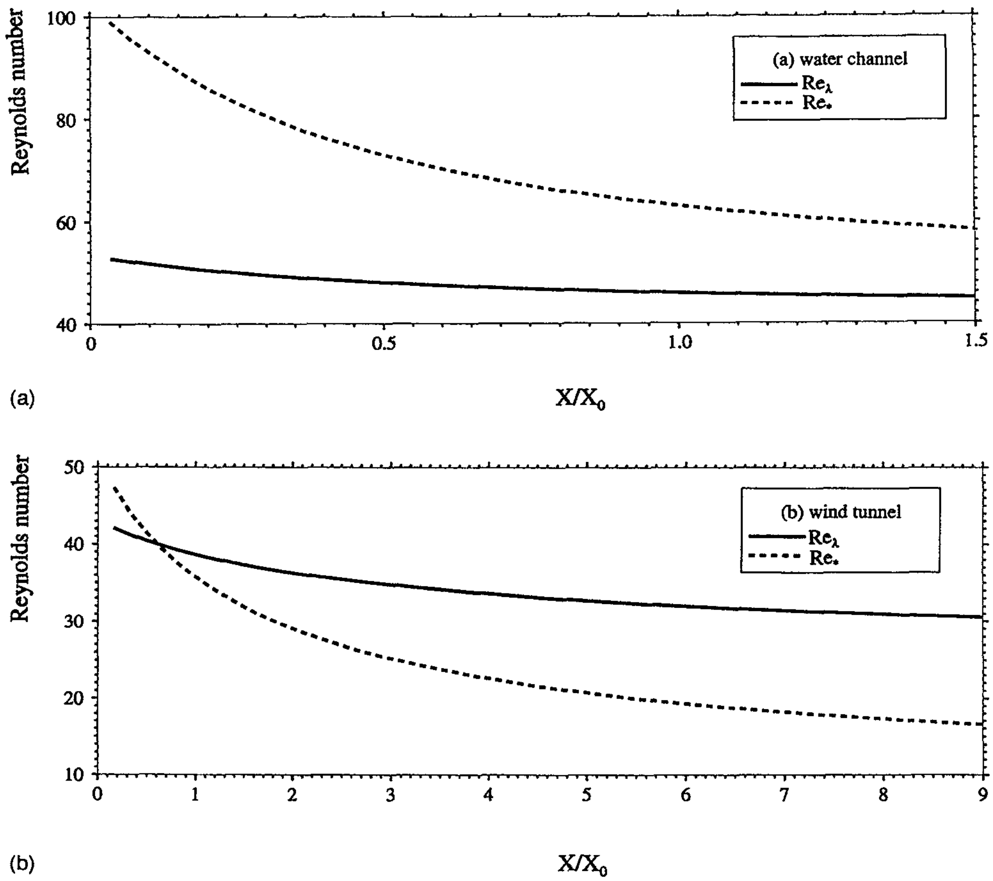

FIG. 4. Reynolds number versus along stream distance in the experimental grid turbulence: (a) water channel, (b) wind tunnel.

tween the discrete-time implementation of the above equation for $d(W/\sigma_w)$ and the model compared by WTK against field data can be traced through Wilson *et al.*[19] Durbin[20] may independently have suggested this model.] Now, Durbin[18] has analyzed this model to show that it implies (in the large time limit $t/T \rightarrow \infty$) a random displacement (or zero-order) model

$$dZ^+ = \sqrt{2K(Z)}d\zeta + \frac{\partial K}{\partial Z}\,dt, \tag{26}$$

where $d\zeta(t)$ is a Wiener process ($d\zeta$ has variance dt), and

$$K = \sigma_W^2 T = \frac{2\sigma_W^4}{C_0 \epsilon} \tag{27}$$

is an effective eddy diffusivity.

Flux-gradient experiments in the horizontally-uniform NSL indicate that the eddy diffusivity is $K = \kappa u_* Z$, where u_* is the friction velocity, and the von Kármán constant (κ) is now generally accepted as having the value $\kappa = 0.4 \pm 0.02$.[21,22] If this (empirical) result is to be matched with Durbin's result (asymptotic eddy diffusion model), we require that

$$C_0 = 2\left(\frac{\sigma_W}{u_*}\right)^4, \tag{28}$$

where we have used the fact that in the NSL, $\epsilon \approx u_*^3/\kappa Z$. Now, since $\sigma_w \approx 1.3 u_*$, we have $C_0 \approx 5.7$. This is the value suggested by Rodean, here deduced by a logic which avoids reference to the Lagrangian timescale (the latter being undefined in the case of inhomogeneous turbulence). The equivalent result for a Lagrangian time scale (albeit difficult of interpretation) was arrived at much earlier.[23]

This is a pleasing theoretical argument. However one would be unduly bold to claim the present generation of LS models as *ultimately* correct, and may expect Thomson's 1987 criteria eventually to be superseded. The above logic does not *guarantee* that the conformance of (properly selected) LS models with atmospheric observations is optimal, when $C_0 = 5.7$. In fact, several workers have found otherwise. For example, Wilson *et al.*[6] found that a better fit to observed dispersion (Project Prairie Grass) is obtained using (in effect) $C_0 \approx 3.1$ (the WTK model was actually couched in terms of a Lagrangian timescale and WTK wrote $\sigma_w/u_* = 1.25$). Earlier, Reid[23] reached the same conclusion in reference to the Porton field data. Findings corresponding to $C_0 \approx 3.1$ exist[24,25] in the context of *Eulerian* dispersion models, for the magnitude of the turbulent Schmidt number (S_c, the ratio of the eddy diffusivity for mass to the eddy viscosity) giving best agreement with observed (field and wind tunnel) dispersion. $C_0 \approx 3.1$ is close to our best guess for C_0 on the basis of our examination of laboratory experiments.

V. FIRST-ORDER LAGRANGIAN STOCHASTIC MODEL

Sawford[9] found that for homogeneous, isotropic, and stationary turbulence, Reynolds number effects in first-order models can be incorporated by replacing the universal constant C_0 with

$$C=C_0(1+\mathrm{Re}_*^{-1/2})^{-1}. \tag{29}$$

Now we ask: Is this correction to the first-order model useful in homogeneous, isotropic, but decaying turbulence?

For such a flow, the one-dimensional first-order model is[1]

$$dW^+=-\left(\frac{C_0\epsilon}{2\sigma_W^2}-\frac{\sigma_W'}{\sigma_W}\right)W^+dt+\sqrt{C_0\epsilon}d\zeta,$$
$$dZ^+=W^+dt. \tag{30}$$

By replacing C_0 in Eq. (30) by C, as given by Eq. (29), and carrying out a simulation with the revised first-order model, we found that Eq. (29) works well, especially for the wind tunnel experiment (Fig. 3). For comparison, we also show the prediction with $C=3.0$.

Figure 4 shows the Reynolds number in the range of interest of the wind tunnel and water channel experiments. This helps to explain why the correction (29) is more significant for the wind tunnel experiment. In the water channel, the Lagrangian Reynolds number Re_* is sufficiently high that the Reynolds-number correction to the first-order model is not large. But in the wind tunnel experiment Re_* is lower, so C is significantly different from its asymptote.

VI. CONCLUSION

The Sawford[9] model has been shown to be implied by a generalized well-mixed constraint, for (hypothetical) homogeneous, isotropic, stationary turbulence, for which the Eulerian joint PDF for the velocity and acceleration is (putatively) Gaussian.

We have extended that model to cover decaying grid turbulence. By comparing measured and modeled dispersion, the universal Kolmogorov constant C_0 is estimated to be 3.0 ± 0.5, substantially different from the result, $C_0=7.0$, obtained by Sawford by comparing modeled dispersion statistics with direct numerical simulation data.[10]

When the Reynolds-number effect is incorporated into the first-order model via the supposedly universal constant C_0, i.e., by replacing C_0 with a variable C [Eq. (29)], the first-order model also gives a very good prediction, suggesting that Sawford's revision of the first-order model for finite-Reynolds-number flow is satisfactory. This is useful, because first-order models are simpler than second-order, and require less Eulerian statistical information on the flow.

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APPENDIX: THE PDF FOR FIXED-POINT ACCELERATION

The spatial derivative of velocity is not Gaussian,[12] and recent studies of isotropic turbulence show that, even if the single-point velocity PDF is identically Gaussian, the distribution of the pressure fluctuation is negatively skewed.[26,27] While it is not clear how these non-Gaussian properties impact the Eulerian acceleration PDF, we believe the latter is non-Gaussian on this and the following evidence.

Recall that we signify Lagrangian quantities by superscript (+). The Eulerian acceleration field

$$\mathbf{A}(\mathbf{X}_0,\ t_0)=\lim_{t-t_0\to 0}\frac{\mathbf{U}^+(\mathbf{X}_0,\ t)-\mathbf{U}(\mathbf{X}_0,\ t_0)}{t-t_0}$$
$$=\lim_{\tau\to 0}\frac{\Delta_\tau\mathbf{U}^+}{\tau} \tag{A1}$$

is defined by the difference of Lagrangian velocity over an infinitesimal time interval.[13] Here $\mathbf{U}^+(\mathbf{X}_0,t)$ is velocity at time t of that fluid element which, at time t_0, was at location $\mathbf{X}_0$. We can therefore infer the distribution for Eulerian acceleration if we know the distribution of the Lagrangian velocity difference, taken over a very short time interval. According to Fig. 15 of Yeung and Pope,[10] derived from DNS of isotropic turbulence, the distribution of Lagrangian velocity difference $\Delta_\tau U^+$ (where U^+ is one component of $\mathbf{U}^+$) is symmetric about $\Delta_\tau U^+=0$ for any time interval τ, and deviates from the Gaussian distribution as τ gets smaller. When τ is extremely small ($\tau\sim t_\eta$), the distribution appears to be exponential. This suggests the PDF for Eulerian acceleration may be exponential, and symmetric about $A=0$.

We derived a second-order model (for stationary turbulence) from the exponential PDF, and compared its prediction for tracer spread with the model of Sec. II. No substantial difference was found: out to $t/T_L=10$, the maximum difference was less than 5% in σ_Z, and had no effect on the choice of $C_0=3.0\pm0.5$. In modeling the mean concentration field, we are concerned with low-order statistics of highly integrated properties (position is twice-integrated acceleration). For this reason, we suggest that for our purposes the adoption of a Gaussian PDF for Eulerian acceleration is acceptable.

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